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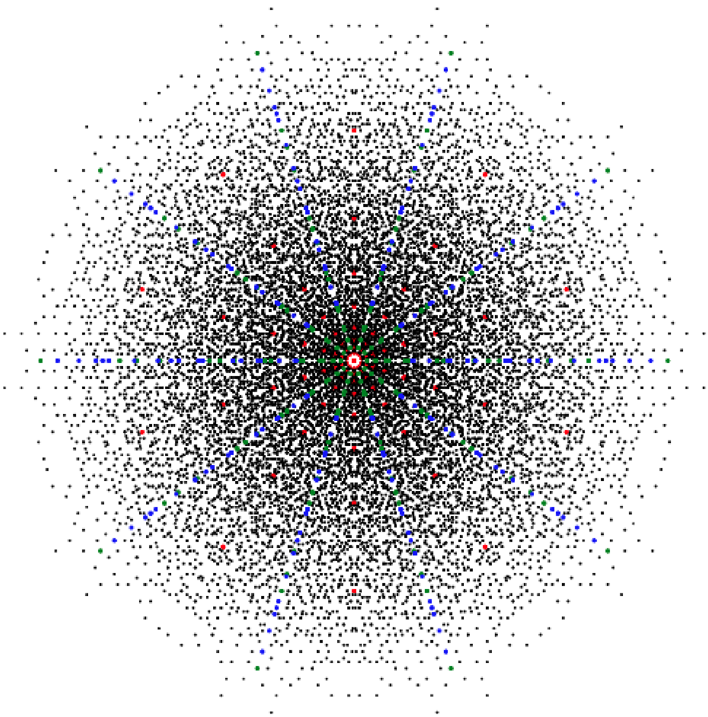
QARCH



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Qarch Issue 1

The questions are *not* arranged in any particular order.

1. Construct a finite group G and a subgroup H such that $|\operatorname{Aut}(H)| > |\operatorname{Aut}(G)|$, where $\operatorname{Aut}(G)$ denotes the group of automorphisms of G. Is this still possible if H is required to be normal?

2. Since $\mathbb{Q}$ is an ordered field, one can define differentiation on $\mathbb{Q}$ in the normal way. Find an uncountable number of solutions to the differential equation $f' = f,\ f(0) = 1$ for $f : \mathbb{Q} \to \mathbb{Q}$.

3. Let α_i be positive real numbers which sum to 1. Let A_i be positive definite matrices. Prove that

$$\det(\alpha_1 A_1 + \cdots + \alpha_n A_n) \geq (\det A_1)^{\alpha_1} \cdots (\det A_n)^{\alpha_n}$$

4. Does there exist a polynomial $p : \mathbb{R}^2 \to \mathbb{R}$ with image $(0, \infty)$?

5. Given a group G, construct a group H containing G such that every automorphism of G is induced by conjugation by an element in H.

6. Let S be a subset of the positive reals of cardinality 4. Prove that there exist $x, y \in S$ such that $(x^2 - 3)(y^2 - 3) > 8(1 - xy)$.

7. A vector v is distributed uniformly and randomly over the surface of the unit sphere in an N-dimensional Euclidean space. What is the expected value of the squared length of the projection of v onto an M-dimensional linear subspace?

8. You play a game flipping a fair coin. You may stop after any trial, at which point you are paid in pounds the percentage of heads flipped. So if on the first trial you flip a head, you should stop and earn £100 because you have 100% heads. If you flip a tail then a head, you could either stop and earn £50, or continue on, hoping the ratio will exceed 1/2. This second strategy is superior.

   Suppose you have flipped 5 heads in 8 trials. Determine whether it is better to continue or stop, i.e. accept £62.50 or continue flipping the coin and hope for more.

9. Let $p(x)$ be a polynomial which only takes non-negative values. Prove that it can be written as a sum of two squares.

10. Let p be prime, and let e, r be positive integers. Prove that for all sufficiently large n, $\sum_m (-1)^m \binom{n}{m}$ is divisible by p^e, where the sum is taken over all m such that $1 \leq m \leq n$ and $m \equiv r \pmod{p}$.

11. Prove that within any set of 16 distinct positive integers not exceeding 100 there are four different elements a, b, c, d, such that $a + b = c + d$.

12. What is the largest number of acute angles in a non-convex n-gon?

13. Let K be an ordered field. Suppose that convergence of series in K satisfies the ratio test condition, prove that K has the least upper bound property (the topology is taken to be the order topology, generated by $\{x : x > a\}$ and $\{x : x < b\}$ for $a, b \in K$).

14. A frog lives on a pond with n lined-up equidistant lily pads. Suppose that the frog wants to visit every lily pad exactly once, but in such a way that it makes no two leaps of the same length. For which n and for which starting positions can the frog carry out its plan?

15. Given a rectangle, for which odd n is it possible to divide it into n congruent non-rectangular polygons?

16. Do there exist two matrices that are conjugate in $\mathrm{SL}(2, \mathbb{Z}/n\mathbb{Z})$ for all natural numbers n but are not conjugate in $\mathrm{SL}(2, \mathbb{Z})$?

17. Prove that any convex polygon of area 1 can be placed inside a rectangle of area 2.

18. Show that for all integers n, there exists a circle which passes through exactly n lattice points (points for which both x and y coordinates are integers).

19. Prove that $\mathbb{R}^n$ is not a countable union of affine subspaces, i.e. translations of proper linear subspaces.

20. Let the sequence a_n be defined by $a_0 = 2$ and $a_n = a_{n-1} \frac{n+a_{n-1}}{n+1}$. What is the least n for which a_n is not an integer?

Comments and corrections to `archim-qarch-editor@srcf.ucam.org`
Copies available at `http://www.archim.org.uk/qarch.php`

The cover picture shows those primes in the ring of integers of $\mathbb{Q}[\zeta]$ which can be written as $\sum_{i=0}^{4} a_i \zeta^i$ with $a_i \in \{0, \pm 1, \pm 2, \pm 3\}$, where $\zeta = e^{\frac{2\pi i}{5}}$ is a 5th root of unity. The factorization of a rational prime p depends on the value of $p \pmod 5$, and this is reflected in the colors.

QARCH is a problems journal published termly by the Archimedeans. Each issue consists of 20 questions of varying difficulties. Comments, full or partial solutions, new questions or queries can be sent to

`archim-qarch-editor@srcf.ucam.org`

There is minimal restriction on the type of question suitable for publication and discussion in these pages, except that the statement should be relatively concise, and that the solution should not be immediately obvious. Problems may be submitted with or without known solutions.

Hints or solutions to selected questions may appear eventually at

`http://www.archim.org.uk/qarch.php`

The following people have helped with the production:

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