Complex Methods 1B Lent 2010

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Contents

1 Introduction

The course is of 16 lectures consisting of four topics of roughly equal duration with emphasis on applications, rather than general theory. Thus the examples are as important as formal theorems. The topics are

- **Analytic (or holomorphic) functions**, the main applications are to

  a) Solving Laplace's equation on $\mathbb{R}^2$, which is essential for electrostatic and fluid dynamical problems.

  b) Conformal Mappings, which may also be used for solving Laplace's equation but which also have applications to many other problems including Cartography.

- **Contour Integrals**

  a) Cauchy's Theorem as an application of Stokes's Theorem on $\mathbb{R}^2$.

  An important application is to

  b) The evaluation of real integrals such as

$$\int_o^\infty \frac{dx}{1+x^6}, \quad \int_o^\infty \frac{dx}{1+x^3} \quad \int_0^{2\pi} \frac{d\theta}{(1+3\cos^2\theta)} \,. \tag{1.1}$$

- **Residue Calculus** which is an algorithm for the evaluation of contour integrals by an examination of the singularities of the integrand (called in this context their *poles*).

- **Fourier and Laplace transformations**. The first should be familiar: it allows one to reduce the solution of ordinary and partial differential equations (O.D.E.'s and P.D.E.'s) to algebra and the evaluation of integrals using the formulae

  a) if

$$\mathcal{F}(f(t)) = \tilde{f}(\omega) := \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \tag{1.2}$$

  with inverse

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{+i\omega t}d\omega \tag{1.3}$$

  The evaluation of either (1.2) or (1.3) is often most conveniently effected using contour integration.

  b) The Laplace transform is defined by

$$\mathcal{L}(f(t)) = \hat{f}(s) := \int_0^{\infty} e^{-st}f(t)\,dt, \qquad s\in\mathbb{C} \tag{1.4}$$

  and the inversion formula analogous to (1.3) is a contour integral.

2 Books

Almost all "mathematical methods "books contain an account of most of the material in this course. One such is

G. Arfken and H. Weber, *Mathematical Methods for Physicists*.
Rather more mathematically detailed is

H Priestley *Introduction to Complex Analysis*.

Two other good books are

I Stewart and B. Tall *Complex Analysis*

and

M J Ablowitz and A.S. Fokas *Complex Variables*.

The latter goes somewhat beyond the course.
While preparing the lectures I made use of, among others,

E.G. Phillips *Functions of a Complex Variable*.

3 Analytic Functions

3.1 Complex numbers and the complex plane

The special flavour of complex analysis arises because one may think of the complex numbers $\mathbb{C}$ both *algebraically* as a number system and *geometrically* as a vector space. It is essential therefore to have a good geometrical intuition for the complex plane and so we shall start by briefly reviewing what should be well known.

Complex numbers z are points in $\mathbb{R}^2$ with coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}$ equipped with a commutative and associative multiplication law written

$$z, z' \to zz' = z'z \tag{3.1}$$

given by

$$(x, y)(x', y') = (xx' - yy', xy' + x'y) \tag{3.2}$$

We may take $1 = (1, 0)$ and $i = (0, 1)$ as a basis for $\mathbb{R}^2$ [1] and write

$$z = x + iy\,, \qquad \text{with} \quad i^2 = -1. \tag{3.3}$$

complex conjugation is reflection in the horizontal axis

$$(z, y) \to (x, -y) = \bar{z} \tag{3.4}$$

[1] In some textbooks for scientists and engineers one sees j used rather than i. Vectors in the complex plane, especially of the form $e^{j\omega t}$, with t being thought of as time are often called *phasors* and represented by arrows which, for positive ω, rotate in an anticlockwise sense as time increases. The complex conjugate of a phasor is an anti-phasor which rotates in a clockwise sense.

such that

$$\overline{zz'} = \bar{z}\,\bar{z}' \quad \text{etc} \tag{3.5}$$

The *norm* or *modulus* of a complex number is defined by

$$|z| := \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \tag{3.6}$$

the *positive* square root being taken. The plane $\mathbb{R}^2$ so equipped is called the *complex plane* and denoted $\mathbb{C}$.

On may introduce *polar coordinates* (r, θ) for $\mathbb{C} \setminus 0$, they consist of the

$$\text{modulus} \qquad r = \sqrt{x^2 + y^2} = |z| \,, \tag{3.7}$$

$$\text{and phase} \qquad \theta = \arg z = \arctan(\frac{y}{x}) \,. \tag{3.8}$$

Obviously, the phase θ is not defined at the origin $(0, 0)$ because the all radial coordinate lines $\theta = \text{constant}$ intersect there. Moreover there is some ambiguity in taking the inverse when defining $\arctan(y/x$ It is also clear, that whatever origin we chose for θ, i.e. whatever radial coordinate line $\theta = \text{constant}$ on which we set $\theta = 0$,

$$\boxed{\theta \text{ is only defined up to addition of an integer multiple of } 2\pi \text{ , unless we fix a convention}} \tag{3.9}$$

The *Principal Value* of the function $\arctan(y/x)$ is defined by

$$-\pi < \arg z < \pi \tag{3.10}$$

We have not defined θ precisely along the negative real axis and it clearly jumps by 2π as we cross the negative real axis. These elementary observations will be important later. In the mean time we turn to

3.2 Complex Valued Functions

A *complex function* or more precisely a *complex valued function* is just a map $g : \mathbb{C} \to \mathbb{C}$, which we may also regard as a map $g : \mathbb{R}^2 \to \mathbb{R}^2$, sending

$$(x, y) \to (u(x, y), v(x, y)) \,, \tag{3.11}$$

$$\text{i.e.} \qquad z \to w = u(x, y) + iv(x, y) = g(z, \bar{z}) \,. \tag{3.12}$$

It is sometimes helpful to think of w as lying in its own “complex w plane ”. In what follows we may need to drop the requirement that g be defined for all $\mathbb{C}$. It may, for example only be defined in an open subset of $\mathbb{C}$. This is done as follows. We have first

Definition *An open disc $D(z_0, R)$ of radius R centred on some point z_0 given by*

$$D(z_0, R) = \{z : \quad |z - z_0| < R\} \,. \tag{3.13}$$

Definition *An open set $U \subset \mathbb{C}$ is a (possibly infinite) union or a finite intersection of open discs.*

Since

$$x = \frac{z+\bar{z}}{2} \qquad y = \frac{z-\bar{z}}{2i}\,, \tag{3.14}$$

an equivalent way of specifying a complex valued function is, as indicated in the second equation of (3.12), to use $z, \bar{z}$ rather than x, y as coordinates for $\mathbb{R}^2$, considered as the domain of the map g and hence to adopt a notation in which g as expressed as a function of z and $\bar{z}$. In other words, within its domain of definition, we write

$$w = u(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}) + iv(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}) := g(z, \bar{z}) \tag{3.15}$$

Example

$$u = x^2 - y^2 + x\,, \qquad v = 2xy - y \tag{3.16}$$

$$g(z, \bar{z}) = z^2 + \bar{z} \tag{3.17}$$

By contrast, if we had considered the function

$$u = x^2 - y^2 + x\,, \qquad v = 2xy + y \tag{3.18}$$

we would have had

$$g(z, \bar{z}) = z^2 + z\,. \tag{3.19}$$

3.3 Analytic or holomorphic functions

We see that the expression $g(z, \bar{z})$ giving g will in general contain both z and $\bar{z}$ but it can happen that terms involving $\bar{z}$ are absent. Such functions are referred to as *analytic* or *holomorphic*. They are also sometime as referred to as *regular*. Intuitively they depend only z but not on $\bar{z}$. We shall give a precise definition shortly. Whatever one calls them, they have many beautiful properties, and from now on we shall mainly work with such functions which we shall write as

$$z \longrightarrow w = f(z). \tag{3.20}$$

Of course there is an obvious notion of *anti-analytic* or *anti-holomorphic* function obtained by interchanging z and $\bar{z}$, One way to make precise the idea that f does not depend on $\bar{z}$ is consider the analogue of the *complex derivative*. Suppose g is defined in a open disc about z_0. We might consider evaluating

$$g'(z_0) = \frac{dg}{dz} = \lim_{h \to 0}\Big(\frac{g(z_0+h) - g(z_0)}{h}\Big) \tag{3.21}$$

The problem is that since h is an infinitesimal 2-vector, as is $g(z_0+h) - g(z_0)$, and we are attempting to divide a vector by a vector and the limit may fail to exist for a variety of reasons e.g.

- g(z) may be ill-defined, for example $g(z) = \frac{1}{z^2}$ at $z_0 = 0$.
- more significantly *the limit may depend on the direction in which we take the limit*, that is it may depend on $\alpha = \arg h$.

Example $g = \bar{z}$

$$\frac{d\bar{z}}{dz} = \lim_{h\to 0} \frac{\bar{h}}{h} = e^{-2i\alpha}\,, \qquad \alpha = \arg h \tag{3.22}$$

which clearly depends on direction, i.e. on α. We would have similar problems with say

$$\frac{d(z\bar{z}^2)}{dz} = \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h})^2 - z\bar{z}^2}{h} \tag{3.23}$$

but not with

$$\frac{d(z^2)}{dz} = \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h\to 0} 2z + h = 2z\,. \tag{3.24}$$

Thus we adopt the following

Definition *$f(z)$ is holomorphic or analytic or differentiable in an open disc $D(z_0, R)$ (or more generally an open set $U \subset \mathbb{C}$) if the derivative f' exists $\forall z \in U$.*

Definition *$f(z)$ is analytic at $z_0 \in \mathbb{C}$ if there exists a disc $D(z_o, R)$ such that $f(z)$ is analytic in $D(z_0, R)$.*

Definition *A function $f(z)$ is said to be singular at z_0 if it not analytic at z_0*

Definition *An entire function $f(z)$ is one which is analytic in the entire complex plane*

Example A polynomial in z is entire . The functions $\sin z$ and $\cos z$ are entire.

It may be proved that if a function is once complex differentiable in a disc $D(z_o, R)$, then it is infinitely complex differentiable. Moreover it has a Taylor series in powers of z, with no $\bar{z}$'s, centred on z_0 with which converges within $D(z_0, R)$. This latter property is often taken as the definition of an analytic function. In fact it may be shown that it implies the definition which we have adopted.

3.4 The Cauchy Riemann Equations

Theorem *A necessary and sufficient condition that $g(z) = u + iv$ to be analytic in $U \subset \mathbb{C}$ is that u and v have continuous first partial derivatives and and such that*

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{3.25}$$

We prove the necessity and omit the sufficiency. If h is real we find that

$$\frac{dg}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\,. \tag{3.26}$$

If h is pure imaginary

$$\frac{dg}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\,. \tag{3.27}$$

Equating real and imaginary parts gives the CR equations (3.25).

We may make contact with our previous intuitive idea that a holomorphic function depends on z but not $\bar{z}$ by defining

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\,, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\,. \tag{3.28}$$

Thus

$$\frac{\partial z}{\partial z} = 1\,, \qquad \frac{\partial z}{\partial \bar{z}} = 0\,, \qquad \frac{\partial \bar{z}}{\partial \bar{z}} = 1\,, \qquad \frac{\partial z}{\partial \bar{z}} = 0 \tag{3.29}$$

We expect that holomorphic function $f(z) = u + iv$ should satisfy

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)(u + iv) = 0\,. \tag{3.30}$$

Equating real and imaginary parts gives (3.25) as expected.

Example Is $u = e^x \cos y$ the real part of an analytic function? If so, what is v?

By the CR equations we must have

$$\frac{\partial v}{\partial y} = e^x \cos y\,, \qquad -\frac{\partial v}{\partial x} = -e^x \sin y \tag{3.31}$$

One checks that mixed partials are equal and integrates

$$v = e^x \sin y + \text{function of } x \tag{3.32}$$

$$v = e^x \sin y + \text{function of } y \tag{3.33}$$

Thus

$$f = e^x(\cos y + i \sin y) = e^{x+iy} + \text{constant} = e^z + \text{constant} \tag{3.34}$$

Example Is $z\bar{z}$ analytic ?

$$u = x^2 + y^2\,, \qquad v = 0\,, \tag{3.35}$$

and the CR equations (3.25) are not satisfied.

3.5 Some Consequences of the Cauchy Riemann equations

Cor.1 *(i) The product gf and (ii) the the composition $r = g \circ f$ of two holomorphic maps is holomorphic*

Both properties are intuitively obvious but we can prove them formally using the CR equations (3.25) .

(i) If $f(z) = u(x,y) + iv(x,y)$ and $g(z) = s(x,y) + it(x,y)$, then

$$gf = (us - vt) + i(vs + ut) \tag{3.36}$$

Leibniz and the CR equations for u, v and s, t show that $(us - vt)$ and $(vs + ut)$ also satisfy the CR equations

(ii) We have

$$w = f(z) = u(x,y) + iv(x,y)\,, \qquad r = g(w) = s(u,v) + it(u,v) \tag{3.37}$$

the composition is

$$r(z) = g \circ f == g(f(z)) \tag{3.38}$$

$$= s(u(x,y), v(x,y)) + it(u(x,u), v(x,y)) \tag{3.39}$$

The chain rule gives

$$\begin{aligned}
\partial_x s &= \partial_u s\, \partial_x u + \partial_v s\, \partial_x v && (3.40)\\
\partial_y t &= \partial_u t\, \partial_y u + \partial_v t\, \partial_y v && (3.41)
\end{aligned}$$

But

$$\begin{aligned}
\partial_u s = \partial_v t\,, \qquad & \partial_v s = -\partial_u t\,, && (3.42)\\
\partial_x u = \partial_y v\,, \qquad & \partial_y u = -\partial_x v && (3.43)
\end{aligned}$$

Thus

$$\partial_x s = \partial_y t \tag{3.44}$$

A similar argument shows that

$$\partial_y s = -\partial_x t \tag{3.45}$$

Cor.2 *The level sets* $u = \text{constant}$ *and* $v = \text{constant}$ *are orthogonal.*

The normals are

$$\nabla u = (\partial_x u, \partial_y u)\,, \qquad \nabla v = (\partial_x v, \partial_y v) \tag{3.46}$$

Thus

$$\begin{aligned}
\nabla u.\nabla v &= \partial_x u \partial_x v + \partial_y u \partial_y v && (3.47)\\
&= \partial_y v \partial_x v - \partial_x v \partial_y v = 0\,, && (3.48)
\end{aligned}$$

by the CR equations (3.25).

Cor.3

$$\boxed{|\nabla u|^2 = |\nabla v|^2 = |\frac{df}{dz}|^2} \tag{3.49}$$

On has by (3.25)

$$|\nabla u|^2 = (\partial_x u)^2 + (\partial_y)^2 = (\partial_x v)^2 + (\partial_y u)^2\,. \tag{3.50}$$

Moreover

$$\begin{aligned}
\frac{df}{dz} &= \frac{1}{2}(\partial_x - i\partial_y)(u + iv) && (3.51)\\
&= \frac{1}{2}\big(\partial_x u + \partial_y v + i(\partial_x v - \partial_y u)\big) && (3.52)\\
&= \partial_x u - i\partial_y u && (3.53)\\
&= \partial_y v + i\partial_x v && (3.54)\\
& && (3.55)
\end{aligned}$$

Cor.4 *The Jacobian* $\frac{\partial(u,v)}{\partial(x,y)}$ *of the map is positive, i.e. the map preserves orientation*

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = v_x^2 + v_y^2 = |\nabla v|^2 = u_x^2 + u_y^2 = |\nabla u|^2 \,. \tag{3.56}$$

Definition A *closed curve* $z = \gamma(t)$ is one for which there is some least value $T > 0$ for which $\gamma(t) = \gamma(t+T)$, $\forall\, t \in \mathbb{R}$. A *simple closed curve* $z = \gamma(t)$ is one for which if $z(t_1) = z(t_2)$ with $|t_2 - t_1| < T$ implies $t_i = t_2$.

That is a simple closed curve does not intersect itself. Thought of as a continuous map from the circle S^1 to the plane $\gamma : S^1 \to \mathbb{R}^2$ it is $1-1$ on its image. We shall often, but not always, chose the parameter t along the curve such that $T = 2\pi$ and restrict the parameter to lie in the interval $0 \le t \le 2\pi$.

The Jordan Curve Theorem States that a simple closed curve γ bounds a domain D topologically equivalent (i.e. continuously deformable) to the unit disc $|z| < 1$ and such that $\gamma = \partial D$ maps to the unit circle $|z| = 1$

In what follows,we shall usually assume that the direction of increasing t is chosen so that the domain D is on one's left hand side as t increases, as it it would if one traversed the unit circle in the direction of increasing θ.

The positivity of the Jacobian (3.56) has the important consequence that if one follows a simple closed curve $\gamma(t) = \partial D$ given say by a complex valued periodic function of t

$$z = z(t) = x(t) + iy(t)\,, \qquad z(t) = z(t+2\pi) \tag{3.57}$$

in the z-plane with the inside D on ones left hand side as t increases . The image $w = w(t) = f(\gamma = f(z(t))$ in the w-plane will be a closed curve $f(\gamma)$ and if it is simple, then the inside $f(D)$ will also be on one's left hand side as t increases.

Example *Elliptical Coordinates in* $\mathbb{R}^2$. Set

$$w = \cosh^{-1} z = \ln(z + \sqrt{z^2-1})\,. \tag{3.58}$$

There is some ambiguity in the definition of the inverse, but let's ignore it for the time being. One has

$$x + iy = \cosh(u + iv) \tag{3.59}$$

$$= \cosh u \cos v + i \sinh u \sin v \tag{3.60}$$

$$x = \cosh u \cos v\,, \qquad y = \sinh u \sin v\,. \tag{3.61}$$

The curves $u = \text{constant}$ and $v = \text{constant}$ are ellipses and hyperbolae

$$\Big(\frac{x}{\cosh u}\Big)^2 + \Big(\frac{y}{\sinh u}\Big)^2 = 1\,. \qquad \Big(\frac{x}{\cos v}\Big)^2 - \Big(\frac{y}{\sin v}\Big)^2 = 1\,. \tag{3.62}$$

Simple geometry shows that all the ellipses and all hyperbolae are not only orthogonal but share the same foci at (± 1). The interval between the foci $(-1 < x < 1)$ corresponds to $u = 0$. At large distances, the hyperbolae $v = const$ approach the radial lines $\theta = v$ and the ellipses the circles $r = \frac{1}{2}e^u$. To cover the (x, y) plane we need $0 \le u < \infty$ and $-\pi < v \le \pi$. The ellipses $u = u_o$ are simple closed curves, parametrized by v. As v increases the exterior $u > u_0$ of an ellipse is on ones right hand side. The ellipses are mapped to vertical intervals $(u_0, \pi \le v \le \pi)$ in the w-plane, and the exterior is mapped to an infinite strip on its right hand side.

3.6 Harmonic Functions on $\mathbb{R}^2$

Definition A *Harmonic function* $\phi(x, y)$ *on* $\mathbb{R}^2$*, or possibly a open subset or domain* $D \subset \mathbb{R}^2$*, is a real valued function satisfying Laplace's equation*

$$\nabla^2\phi = \partial_x^2\phi + \partial_y^2\phi = 0\,. \tag{3.63}$$

Clearly we need ϕ to be suitably smooth. Having continuous second partial derivatives is certainly sufficient for our definition to make sense. We have the following important

Proposition *The real and imaginary parts of a function* $f(z)$ *which is analytic in* D *are harmonic. Conversely, given a harmonic function* $\phi(x, y)$ *, there is a so-called conjugate harmonic function* $\psi(x.y)$ *with orthogonal level sets such that* $f = \phi + i\psi$ *is analytic in* D*.*

The proof in one direction is a straight forward consequence of the CR equations(3.25) and the equality of mixed partials. One has

$$\partial_x^2 u = \partial_x\partial_y v = \partial_y\partial_x v = -\partial_y^2 u, \tag{3.64}$$

Similarly for v. Alternatively we have

$$\frac{\partial}{\partial\bar{z}} f(z) = 0 \tag{3.65}$$

Thus

$$\frac{\partial^2}{\partial z\partial\bar{z}} f(z) = \frac{1}{4}(\partial_x - i\partial_y)(\partial_x + i\partial_y)f = \frac{1}{4}\big(\partial_x^2 + \partial_y^2 - i\partial_y\partial_x + i\partial_x\partial_y\big)f = \frac{1}{4}\nabla^2 f = 0\,. \tag{3.66}$$

Now take real and imaginary parts of (3.66) . *Conversely* given a harmonic function $\phi(x, u)$, set

$$f(z) = \phi + i\psi \tag{3.67}$$

for some $\psi(x, y)$ to be found, and impose the CR equations (3.25)

$$\partial_x\psi = -\partial_y\phi \qquad \partial_y\psi = \partial_x\phi \tag{3.68}$$

The integrability condition for the exact differential

$$d\psi = -\phi_y dx + \phi_x dy \tag{3.69}$$

is precisely Laplace's equation (3.66). *Alternatively*, we can use the notation of vector analysis. We have to solve

$$\nabla\psi = \mathbf{A} \tag{3.70}$$

where

$$\mathbf{A} = (-\partial_y\phi, \partial_x\phi, 0) \tag{3.71}$$

The integrability condition is

$$\begin{aligned}
\text{curl}\,\mathbf{A} &= \nabla \times \mathbf{A} && (3.72)\\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -\partial_y\phi & \partial_x\phi & 0 \end{vmatrix} && (3.73)\\
&= (0, 0, \partial_x^2\phi + \partial_y^2\phi) = 0\,. && (3.74)
\end{aligned}$$

Example
$$e^{\frac{z}{a}} = e^{\frac{x}{a}}\cos(\frac{y}{a}) + ie^{\frac{x}{a}}\sin(\frac{y}{a})\,, \qquad a \in \mathbb{R} \tag{3.75}$$

These two solutions , together others obtained from $e^{-\frac{z}{a}}$ and $e^{\pm i\frac{z}{a}}$ give the solutions one would obtain if one separates variables, i.e. makes the *ansatz*

$$\phi(x, y) = g(x)h(y) \tag{3.76}$$

Example *Electrostatics in the plane.* Since $\mathrm{curl}\,\mathbf{E} = \nabla \times \mathbf{E} = 0$, we have $\mathbf{E} = -\nabla\phi$ where ϕ is the electrostatic potential. Since $\mathrm{div}\,\mathbf{E} = \nabla.\mathbf{E} = 0$ we have ϕ is harmonic $\nabla^2\phi = 0$. The curves $\phi = \mathrm{constant}$ are *isopotentials* and the orthogonal trajectories $\psi = \mathrm{constant}$ are *electric field lines*. An electric charge may rest in equilibrium at a point at which $\mathbf{E} = 0$, i.e at a *critical point* of ϕ at which $\partial_x\phi = 0 = \partial_y\phi$. By the Cauchy Riemann equations this is also a critical point of the conjugate function ψ, at which $\partial_x\psi = 0 = \partial_y\psi$. Assuming that it is non-vanishing, the stability is governed by the *Hessian* i.e. the matrix of second partial derivatives

$$\begin{pmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{pmatrix} \tag{3.77}$$

Since ϕ solves Laplace's equation (3.66) the Hessian matrix is trace free. This implies that if the eigen values are both non-vanishing, then they cannot have the same sign. In other words the critical point is a *saddle point* and the equilibrium must therefore be unstable. This result is called *Earnshaw's Theorem*. It also applies to the conjugate function ψ which also has a saddle point at the same position.

Example $f(z) = z^2 = z^2 - y^2 + 2ixy$ has a saddle point at the origin. The isopotentials and field lines make up two orthogonal families of of rectangular hyperbolae, the asymptotes of one system being orthogonal to the asymptotes of the other.

In fact this is the general behaviour of an analytic function in the vicinity of a point where the derivative vanishes, $\frac{df}{dz} = 0$, but the second derivative is non-vanishing $\frac{d^2f}{dz^2} \neq 0$.

3.7 Multi-valued functions, branch points and branch cuts

Most holomorphic functions one encounters are not *entire* , i.e. globally defined and holomorphic in the entire complex plane, merely in some subset. In fact one frequently defines a function locally in some domain D and then seeks to find a larger, possible the largest, domain $D' \supset D$ in which it is well defined, single valued and holomorphic.

Example $f = z^{\frac{1}{2}}\,, \qquad \ln z\,, \qquad z^\alpha\,, \qquad \alpha \notin \mathbb{Z}.$

Let's start with $z^{\frac{1}{2}}$. There is an obvious $\pm$ ambiguity.In Cartesian coordinates we have

$$u^2 - v^2 = x\,, \qquad 2uv = y\,, \tag{3.78}$$

whose solution is

$$u^2 = \frac{x \pm \sqrt{x^2 + y^2}}{2}\,, \qquad v^2 = \frac{\pm\sqrt{x^2 + y^2} - x}{2} \tag{3.79}$$

We may fix this $\pm$ ambiguity by demanding that u^2 and v^2 are real and positive

$$u^2 = \frac{x + \sqrt{x^2 + y^2}}{2}, \qquad v^2 = \frac{\sqrt{x^2 + y^2} - x}{2} \tag{3.80}$$

We thus have

$$u = \pm\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}}, \qquad v = \pm\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} \tag{3.81}$$

There is apparently a choice of four possible sign combinations but from the second equation of (3.78), if $y > 0$, the two signs must be taken to be the same, and if $y < 0$ they must be taken to be opposite.

We can fix the sign of u so that the square root is positive on the real axis. This gives what is called one *branch* of the function $z^{\frac{1}{2}}$. If we decide that $z^{\frac{1}{2}}$ is negative on the real axis we get the other branch.

This, slightly complicated, situation can be simplified by passing to polar coordinates. We have

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\frac{1}{2} i\theta} \tag{3.82}$$

Since, however we define θ, we cannot fix it globally to better than the addition of an integral multiple of 2π, we cannot expect that $z^{\frac{1}{2}}$ to be globally better defined than up to a factor of ± 1. If we adopt what we called earlier the *principal branch* for θ

$$-\pi < \theta < \pi \tag{3.83}$$

we have

$$-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2} \tag{3.84}$$

This $z^{\frac{1}{2}} = i$ just above the negative real axis and $z^{\frac{1}{2}} = -i$ just below the real axis. In order to obtain a domain D in which $z^{\frac{1}{2}}$ is single valued and holomorphic we *cut or slit the complex plane along the negative real axis* $(x \leq, 0)$. That is, we omit the negative real axis, and choose the domain $D' = \mathbb{C} \setminus (x \leq 0, 0)$. The negative real axis is referred to as a *branch cut* , and it includes, and ends on, the origin [2] , which is referred to as a *branch point*. A formal definition will be given shortly.

The function so-defined maps the cut z-plane onto the right hand w-plane, i.e onto $u = \Re w > 0$. This is equivalent to using the upper sign for v in (3.81) when y is positive and the lower sign when y is negative.

If we had chosen the lower sign in (3.81) when y is positive and the upper sign when y is negative, we should have obtained a different branch of the function $z^{\frac{1}{2}}$ which maps the cut z-plane to the left hand w-plane,i.e. to $u = \Re w < 0$.

Example $\log z$. We introduce polars:

$$\log z = \log r + i\theta. \tag{3.85}$$

There are clearly infinitely many possible branches of the logarithm function. If we cut the z-plane along the negative real axis and take the principal branch for θ we get a map of $D' = \mathbb{C} \setminus (x \leq 0, 0)$ into the strip $(-\infty < u < \infty, -\pi < \theta < \pi)$. The other branches map $\mathbb{C} \setminus (x \leq 0, 0)$ into parallel strips displaced vertically by integer multiples of 2π.

[2]i.e. it is the non-positive real axis

Example z^α. We introduce polars

$$z^\alpha = r^\alpha e^{i\alpha\theta} \tag{3.86}$$

and cut as before. If α is rational $\alpha = p/q$ with p and q relatively prime, there will be q branches. Otherwise there will be infinitely many.

Example

$$(z^2-1)^{\frac{1}{2}} = (r_1 r_2)^{\frac{1}{2}} \exp i(\frac{\theta_1+\theta_2}{2}) \tag{3.87}$$

with $z-1 = r_1 e^{i\theta_1}$, $z+1 = r_2 e^{i\theta_2}$.

Consider what happens if we move in the complex plane around a simple closed curve. Now

- $e^{i\frac{\theta_1}{2}}$ is 2-valued if we encircle $z=1$.
- $e^{i\frac{\theta_2}{2}}$ is 1-valued if we encircle $z=1$
- $e^{i\frac{\theta_1}{2}}$ is 1-valued if we encircle $z=-1$
- $e^{i\frac{\theta_2}{2}}$ is 2-valued if we encircle $z=-1$
- $e^{i(\frac{\theta_1+\theta_2}{2})}$ is single valued if we encircle both $z=1$ and $z=-1$ or neither.

We have at least two options for cutting. One is to cut from $z=-1$ to $z=+1$ along the real axis. The other is to introduce two cuts from, one from $-\infty$ to -1 along the real axis, and the other from 1 to $+\infty$ along the real axis. In both cases there at two branch points at $z=\pm 1$. Let's take the first option, slitting the plane from -1 to $+1$ along the real axis. If we than take the principal branches for θ_1 and θ_2. $(z^2-1)^{\frac{1}{2}}$ will then be real and positive on the positive real axis and real and negative on the negative real axis. Just above the cut we have $(z^2-1)^{\frac{1}{2}} = i\sqrt{r_1 r_2}$ and just below the cut $(z^2-1)^{\frac{1}{2}} = -i\sqrt{r_1 r_2}$. The function $(z^2-1)^{\frac{1}{2}}$ so defined has a discontinuity of $2i\sqrt{r_1 r_2}$across the cut (moving downwards).

To give a definition of a branch point we need to introduce the notion of a simply connected domain

Definition An open domain $D \subset \mathbb{C}$ is said to be *simply connected* if every continuous closed curve $\gamma \subset D$ can be continuously shrunk to a point, through a family of curves lying entirely within D.

Definition We say z_0 is not a branch point of a function $f(z)$ if there exists a simply connected domain D containing z_0 such that its restriction $f_{|\gamma} = f(\gamma(t))$ to any closed curve $\gamma(t) \subset D$ is single valued. Otherwise we say that z_0 is a branch point.

Definition We say that $f(z)$ has a branch point at infinity if it is not single valued around all sufficiently large curves.

Example $(z^2-1)^{\frac{1}{2}}$ does not have a branch point at infinity

Example $(z^3-1)^{\frac{1}{2}}$ has three branch points at $z = z = e^{i\frac{2\partial}{3}}$ and $z = e^{-i\frac{2\partial}{3}}$. It also has a branch point at infinity.

While there is no ambiguity in locating branch points, there is some arbitrariness in selecting a set of branch cuts as we have seen. The criterion for a successful choice is that once they have been removed, the resultant function is single valued, at the expense of being discontinuous across the cuts.

Example *The International Date Line*

To understand this, and to clarify what is happening at infinity, it is convenient to consider

3.8 Stereographic Projection and the Riemann Sphere

Consider a unit sphere S^2 in 3-dimensional Euclidean space given say by

$$\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2 = 1\,, \tag{3.88}$$

with its south pole (SP) (i.e. $\mathbf{x} = (0, 0, -1)$) resting on (i.e. tangent to) the plane Π given by $x_3 = -1$. We may map all points $\mathbf{x} \in S^2$ except the north pole (NP) (i.e. $\mathbf{x} = (0, 0, +1)$). onto the plane by continuing the straight line from the north pole through the point $\mathbf{x}$ onto the plane Π. If ϕ is azimuth (i.e. angle of rotation about the diameter joining the NP and SP) and β co-latitude (i,e. the angle measured from the NP) then with a suitable choice of origin and scale the image of $\mathbf{x} = (\sin\beta\cos\phi, \sin\beta\sin\phi, \cos\beta)$ [3] on the plane Π is given by

$$z = x + iy = e^{i\phi}\cot(\frac{\beta}{2})\,. \tag{3.89}$$

Thus $\phi = \theta$ modulo integer multiples of 2π.

Definition The map $S^2 \setminus \mathrm{NP} \rightarrow \mathbb{C}$ is called *stereographic projection from the north pole*

Intuitively the NP pole maps to infinity in the complex plane $\mathbb{C}$ and we can think of the sphere as the disjoint union of the plane Π and a point at infinity ∞

$$S^2 = \Pi \sqcup \infty \tag{3.90}$$

We could just as well have considered the plane $\tilde{\Pi}$ given by $x_3 = +1$ tangent to the north pole. Stereographic projection from the south pole takes $S^2 \setminus \mathrm{SP}$ to $\tilde{\Pi}$. Thus

$$S^2 = \tilde{\Pi} \sqcup \tilde{\infty} \tag{3.91}$$

If we introduce a complex coordinate $\tilde{z}$ on $\tilde{\Pi}$ then for a suitable choice of origin and scale we have

$$\tilde{z} = \frac{1}{z} \tag{3.92}$$

Now consider $\Pi \times \tilde{\Pi}$ with coordinates $(z, \tilde{z})$. If we call Γ the map $\mathbb{C} \setminus 0 \rightarrow \mathbb{C} \setminus 0$ given by (3.92) we see that

$$S^2 = \mathbb{C} \times \mathbb{C}/\Gamma \tag{3.93}$$

Note that

$$\mathrm{SP} \;\equiv\; z = 0 \equiv \tilde{z} = \infty\,, \tag{3.94}$$
$$\mathrm{SP} \;\equiv\; \tilde{z} = 0 \equiv\; z = \infty\,. \tag{3.95}$$

Definition

The resultant completion of the complex numbers is called the *Riemann Sphere*

[3] In many books (ϕ, θ) are used for azimuth and co-latitude. Since θ is standard for plane polars, we don't have that option.

Example The *antipodal map*

$$\phi \to -\phi\,, \qquad \beta \to \pi - \beta \tag{3.96}$$

or

$$z \to -\frac{1}{\bar{z}} \tag{3.97}$$

is anti-holomorphic, orientation reversing, and fixed point free.

The long and short of this discussion is that to check for branch points at infinity we use (3.92) to introduce the coordinate $\tilde{z}$ and examine the point $\tilde{z} = 0$

Example

$$f(z) = (z^3 - 1)^{\frac{1}{2}} = \tilde{z}^{-\frac{3}{2}}(1 - \tilde{z}^3)^{\frac{1}{2}} \tag{3.98}$$

There is a branch point at $\tilde{z} = 0$, i.e. at infinity. as well as at $z = 1$ and $z = e^{\pm i\frac{2\pi}{3}}$, i.e. $\tilde{z} = 1$ and $\tilde{z} = e^{\mp i\frac{2\pi}{3}}$

Branch cuts can be taken between $z = 1$ and $z = \infty$ along the real axis and between $z = e^{\pm i\frac{2\pi}{3}}$.

Example

$$f = z^{\frac{1}{2}} = \frac{1}{\tilde{z}^{\frac{1}{2}}} \tag{3.99}$$

This has a branch point at both the north and the south pole. We can run a branch cut along a meridian from pole to pole.

Example

$$f = (z^2 - 1)^{\frac{1}{2}} = \frac{1}{\tilde{z}}(1 - \tilde{z}^2)^{\frac{1}{2}} \tag{3.100}$$

Since $f \to \frac{1}{\tilde{z}}$ at infinity (i.e for small $\tilde{z}$) , f has a singularity at infinity (in fact a pole) but it is *not* a branch point.

Example The International Dateline

If $z = e^{i\theta}\cot(\frac{\beta}{2})$ we let $\theta = 0$ be the Greenwich meridian, then local time is given by

$$\Im \log z = \Re w\,, w = -\log z \tag{3.101}$$

and increase as we go eastward and decreases as we go westward. There are branch points at the north and south pole and by international convention a branch cut is drawn connecting the two. A glance at an atlas will reveal that although this is a perfectly respectable branch cut it is not, for very practical reasons, always along the meridian $\phi = \theta = 180°$. [4].

Definition

The function

$$w = f(z) = -i\log z \tag{3.102}$$

mapping $S^6 \setminus NP \sqcup SP$ to the infinite strip $-\pi u \leq \pi$, $-\infty < v \leq \infty$ is called *Mercator's projection*.

[4]see http://www.phys.uu.nl/~vgent/idl/idl.htm for details

3.9 Riemann Surfaces

Rather than deal with a function with many branches,$f_i(z)$ in other words a collection of functions $f_i(z)$, $i = 1, 2, \ldots, n$ defined on n identical domains, U_i , whose boundaries ∂U_i consist of a set of branch cuts, it is sometimes more convenient to think of a single function defined on a single domain $\Sigma = \overline{\sqcup_{i=1}^{i=n} U_i}$ where the overline indicates that the domains U are glued together across their common boundary branch cuts.

This can give rise to a topologically complicated object, especially if one adds in the points at infinity. The resultant construction is called an *n-sheeted Riemann surface*, the domains or cut planes U_i being the *sheets*. A detailed discussion is beyond the scope of this course.

One way to think of a Riemann surface, is to consider the subset

$$(z, w) = (z, f(z)) \tag{3.103}$$

of $\mathbb{C} \times \mathbb{C} \equiv \mathbb{R}^4$ as z varies over the complex plane or if we add the point at infinity over the Riemann sphere. One obtains in this way a two real dimensional surface or manifold in four-dimensional Euclidean space. It is a pleasing exercise (but far beyond the scope of this course) to show that this surface is like a soap film: it extremizes surface area.

3.10 Conformal Mappings

A general linear map $\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R} \tag{3.104}$$

$$\Longrightarrow w \ = w \ \ \frac{1}{2}(\alpha + i\gamma - i\beta + \delta)z + \frac{1}{2}(\alpha + i\gamma + i\beta - \delta)\bar{z} \tag{3.105}$$

is not holomorphic. For example if $\beta = \gamma = 0$, we get a diagonal matrix which expands the x coordinate by a factor α and the y coordinate by a factor δ. If $\alpha \neq \delta$ this will change the angles that lines through the origin make with the axes and with each other. Circles are taken to to ellipses. Squares with sides parallel to the axes are taken into rectangles with sides parallel to the axes, but a rectangle or even a square whose sides are not parallel to the axes will be taken to a general parallelepiped. Such transformations are called *shears*. By contrast a linear holomorphic map is just the composition of a rotation and a dilatation and thus preserves angles, and takes circles to circles, and rectangles to rectangles.

For a general map we need to consider infinitesimal rectangles, or better the angles between curves.

Definition The tangent vector T of a curve $z = \gamma(t)$ in the z-plane is

$$T = \frac{d\gamma}{dt} = \frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} = \dot{x} + i\dot{y} \tag{3.106}$$

Definition The length of the tangent vector is

$$|T| = \sqrt{\dot{x}^2 + \dot{y}^2} = |\frac{d\gamma}{dt}| \tag{3.107}$$

Definition

Given two such curves,γ_1 and γ_2 , the angle α between them is given by

$$e^{i\alpha} = \frac{T_1}{T_2}\frac{|T_2|}{|T_1|} \tag{3.108}$$

Now consider a general holomorphic map from an open set in the z-plane into an open set in the the w -plane

$$w = f(z) \tag{3.109}$$

It will map a curve $z = \gamma(t)$ in the z-plane to a curve $w = \tilde{\gamma}(t) = f(\gamma(t))$ in the w-plane. The tangent vector is

$$\tilde{T} = \frac{d\tilde{\gamma}}{dt} = \frac{dw}{dt} = \frac{df}{dz}\frac{dz}{dt} = \frac{df}{dz}T \tag{3.110}$$

The angle between the two curves $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(t)$ is given by

$$e^{i\tilde{\alpha}} = \frac{\tilde{T}_1}{\tilde{T}_2}\frac{|\tilde{T}_2|}{|\tilde{T}_1|} = e^{i\alpha}\,. \tag{3.111}$$

Definition

A mapping from an open subset of $\mathbb{R}^2$ to an open subset of $\mathbb{R}^2$ which preserves the angles between curves is is called *conformal*.

From (3.110) it follows that the effect of an analytic mapping on a infinitesimal vector at z in the z-plane is to take it to an infinitesimal vector at $w = f(z)$ in the w- plane which is *magnified* by an amount $|f'|$ and rotated through an angle $\arg f'$. Thus an infinitesimal rectangle of sides dx and dy is taken into and infinitesimal rectangle of sides du and dv which is both magnified and rotated, but it remains a rectangle. If we think of dw as the infinitesimal displacement resulting from an infinitesimal displacement dz we have

$$dw = f'(z)dz \tag{3.112}$$

If the map were not conformal, then an infinitesimal rectangle would be mapped into a infinitesimal parallelepiped, that is it would suffer a *shear*.

3.11 The line element

Pythagoras's theorem tells us that the the infinitesimal distance ds [5] between (x, y) and $(x + dx, y + dy)$ is given by

$$ds^2 = dx^2 + dy^2 = dzd\bar{z} \tag{3.113}$$

In general curvilinear coordinates (u, v) say we have

$$d^2s = E(u, v)du^2 + 2F(u, v)dvdu + G(u, v)dv^2 \tag{3.114}$$

[5]One can of course easily avoid the use of the language of infinitesimals if one wishes, but it simplifies notation considerably,is intuitively clear, and universally used

for some functions E, F, G which are often assembled into a symmetric matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{3.115}$$

and referred to as a "metric tensor ". The expression (3.114) is referred to as *the line element*. The angle α between the coordinate lines is given by

$$\cos\alpha = \frac{F}{\sqrt{EG}} \tag{3.116}$$

The expression (3.114) also holds for the infinitesimal distance on a curved surface. It can be used not only to work out distances but angles and areas as well.

In the case of a holomorphic mapping $w = f(z)$ we have

$$dwd\bar{w} = |f'(z)|^2 dzd\bar{z} \tag{3.117}$$

thus all infinitesimal lengths in w-plane are scaled by a factor $|f'|$ as we saw above.

Example *Elliptical coordinates in the plane* From(3.58) $w = f(z) = \cosh^{-1} z$

$$ds^2 = dzd\bar{z} = |\sinh w|^{-2} 2dwd\bar{w} = \frac{1}{\cosh^2 u - \cos^2 v}\big(du^2 + dv^2\big)\,. \tag{3.118}$$

Example *The Sphere* The line element for the standard unit sphere $S^2 : \mathbf{x}(\beta,\phi) = (\sin\beta\cos\phi, \sin\beta\sin\phi, \cos\beta$ is given by

$$ds^2 = d\mathbf{x}.d\mathbf{x} = d\beta^2 + \sin^2\beta d\phi^2\,. \tag{3.119}$$

Note that the coordinate lines $\phi = \mathrm{constant}$, the *meridians* and the coordinate lines $\beta = \mathrm{constant}$ *the circles of latitude* are orthogonal.

Using the formula for stereographic projection we have

$$\frac{4dzd\bar{z}}{(1+z\bar{z})^2} = d\beta^2 + \sin^2\beta d\phi^2\,. \tag{3.120}$$

From this one deduces

Proposition *Stereographic projection is conformal*

Moreover the composition of angle preserving maps is angle preserving, and we deduce from (3.102) that

Proposition *Mercator's projection is conformal*

In Mercator's w-plane, a straight line makes a constant angle with the the vertical lines $u = \mathrm{constant}$ But these are are the images of the meridians.

Definition A *rhumb line* or *loxodrome* on the sphere is a curve making a constant angle with the meridians.

Thus

Proposition Rhumb lines map to straight lines under Mercator's projection.

Definition A circle on the sphere is the intersection of the sphere with a plane. If the plane passes through the centre is is called *great circle*, otherwise a *small circle*. One has the following

Proposition *Stereographic projection maps circles to circles.*

Recall that if β, ϕ are polar coordinates, the unit sphere in Euclidean space is given

$$x_1^2 + x_2^2 + x_3^2 = 1\,, \tag{3.121}$$

$$x_+ i x_2 = \sin\beta e^{i\phi}\,, \qquad x_3 = \cos\beta\,. \tag{3.122}$$

A plane with unit normal $\mathbf{n}$ is given by

$$x_1 n_1 + x_2 n_2 + n_3 x_3 = p\,, \tag{3.123}$$

and will intersect the sphere provided $p^2 \le 1$. If $n = n_i + i n_2$, then simple calculation starting from the formula for stereographic projection (3.89) converts (3.123) to the form

$$\left| z + \frac{n}{n_3 - p} \right|^2 = \frac{1 - p^2}{(n_3 - p)^2}\,. \tag{3.124}$$

Now (3.124) is the formula for a circle of centre $-\frac{n}{n_3 - p}$ and radius $\frac{\sqrt{1-p^2}}{|n_3 - p|}$. If $n_3 = p$ the plane (3.123) contains NP and and the projection is the straight line

$$zn + \bar{z}\bar{n} = 2p\,. \tag{3.125}$$

It follows that if two circles, either of which may great or small, intersect on the sphere with a certain angle, their stereographic projections will be circles or straight lines which intersect at the same angle. These fact were well known to Hipparchus who established then using classical Euclidean geometry. They are at the basis of all subsequent applications of stereographic to astronomy, crystallography, geology etc, etc.

3.12 The Moebius Map

This is a 1-1 map of the Riemann sphere to itself given by

$$z \to w = \frac{az + b}{cz + d}\,, \qquad ad - bc \neq 0\,. \tag{3.126}$$

Note that a, b, c, d and $\lambda a, \lambda b, \lambda c, \lambda d$ $\lambda \neq 0$ give the same Moebius transformation and so one often imposes the condition

$$ad - bc = 1\,. \tag{3.127}$$

This doesn't fix a, b, c, d completely because a, b, c, d and $-a, -b, -c, -d$ both satisfy (3.127) but give the the same Moebius transformation. However the condition (3.127) does show that there is a six real or three complex parameter's worth of Moebius transformations. As a consequence we have

Proposition *Any three distinct assigned points z_1, z_2, z_3 may be mapped to any other three disinct assigned points w_1, w_2, w_3 by a Moebius transformation.* Since Moebius transformations form a group it is sufficient to take $w_1, w_2, w_3 = 1, 0, \infty$ by

$$w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} . \tag{3.128}$$

If $w_1, w_2, w_3 \neq 1, 0, \infty$ we then compose with the inverse

$$w = \frac{zw_3(w_2 - w_3) + w_1(w_3 - w_2)}{z(w_2 - w_1) + (w_3 - w_2)} . \tag{3.129}$$

which takes $0, 1, \infty$ to w_1, w_2, w_3

Example *Examples of Moebius transformations*

$$\begin{aligned}
w &= e^{i\alpha} z \qquad \alpha \in \mathbb{R} \qquad \text{a rotation} && (3.130)\\
w &= kz \qquad k \in \mathbb{R} \qquad \text{a dilation} && (3.131)\\
w &= z + a \qquad k \in \mathbb{C} \qquad \text{a translation} && (3.132)\\
w &= az + b \qquad a, b \in \mathbb{C}\,, \qquad \text{a general affine map} && (3.133)\\
w &= 1/z \qquad \text{an inversion} && (3.134)
\end{aligned}$$

Now from (3.126)

$$w = \frac{a}{c} + \frac{bc - ad}{a} \frac{1}{cz + d} , \qquad c \neq 0 \tag{3.135}$$

let

$$\begin{aligned}
f_1(z) &= cz + d \qquad \text{a shear} - \text{free affine map} && (3.136)\\
f_2(z) &= \frac{1}{z} \qquad \text{an inversion} && (3.137)\\
f_3(z) &= \frac{a}{c} + \frac{bc - ad}{c} z \qquad \text{a shear} - \text{free affine map} && (3.138)\\
& && (3.139)
\end{aligned}$$

Hence

$$w = f_3(f_2(f_1(z))) = f_3 \circ f_2 \circ f_1(z) = \frac{az + b}{cz + d} . \tag{3.140}$$

Thus we have the following extremely useful

Proposition *Any Moebius transformation can be obtained by composing in order a shear -free affine map, an inversion and another shear free affine map.*

Now shear-free affine maps take circles to circles. What about inversions? A general circle is of the form

$$\begin{aligned}
A(x^2 + y^2) + Bx + Cy + D &= 0\,, \qquad A, B, C, D \in \mathbb{R} \qquad \text{i.e.} && (3.141)\\
Ar^2 + r(B\cos\theta + C\sin\theta) + D &= 0\,. && (3.142)
\end{aligned}$$

If we introduce polar coordinates in the w-plane $w = \rho e^{i\alpha} = \frac{1}{z}$ this becomes

$$A + \rho(B\cos\alpha - C\sin\alpha) + D\rho^2 = 0\,, \qquad i.e. \tag{3.143}$$
$$A + (Bu - Cv) + D(u^2 + v^2) = 0\,. \tag{3.144}$$

Therefore we have, taking into account degenerate cases, the

Proposition *Moebius transformations map circles and straight lines to circles and straight lines.*

Example i) $w = \frac{z-1}{z+1}$ maps $\Re z \geq 0$ to $|w| < 1$, (ii) inversion maps $\Re z \geq \frac{1}{2}$ to $|w-1|^2 \leq 1$.

3.13 *Moebius transformations as Lorentz transformations*

What was not known to Hipparchus, but which plays a big role in in Modern Physics is that the group of Moebius transformations and Lorentz transformations are the same thing. To see why, consider $GL(2,\mathbb{C})$ acting on $\mathbb{C}^2$:

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \tag{3.145}$$

If $z = \frac{Z_1}{Z_2}$ this reproduces a Moebius transformation

$$z \rightarrow \frac{az+b}{cz+d}\,. \tag{3.146}$$

The condition (3.127) implies that the matrix $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in $SL(2,\mathbb{C})$, i.e. $\det S = 1$. However S and $-S$ give the same Moebius transformation and so the group of Moebius transformations may be identified with the group $PSL(2,\mathbb{C}) \equiv SL(2,\mathbb{C})/\pm 1$, where ± 1 is the centre of $SL(2,\mathbb{C})$.

An event in Minkowski spacetime may be assigned coordinates t, x_1, x_2, x_3 and the Lorentz group is by definition the subgroup of $GL(4,\mathbb{R})$ preserving the quadratic form

$$t^2 - x_1^2 + x_2^2 + x_3^2\,. \tag{3.147}$$

Associate to every such event the Hermitian matrix $X = X^\dagger$ given by

$$X = \begin{pmatrix} t + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & t - x_3 \end{pmatrix}. \tag{3.148}$$

and consider the action of $SL(2,\mathbb{C})$

$$X \rightarrow X' = SXS^\dagger \tag{3.149}$$

It preserves the Hermiticity condition $X' = X'^\dagger$ and so takes an event in Minkowski spacetime to an event in Minkowski spacetime. Moreover it preserves the determinant

$$\det X = \det X' \qquad \Leftrightarrow t^2 - x_1^2 - x_2^2 - x_3^2 = t'^2 - x'^2_1 - x'^2_2 - x'^2_3\,. \tag{3.150}$$

Thus every element of $SL(2,\mathbb{C})$ induces a Lorentz transformation. It follows from (3.149) that S and $-S$ induce the same Lorentz transformation. A more elaborate argument, which we will not give here, shows that every Lorentz transformation which preserves space and time orientation may be obtained by a unique Moebius transformation.

Let us return to (3.145) which we write as

$$Z \to SZ \,. \tag{3.151}$$

If

$$Z = \begin{pmatrix} \sqrt{2}\cos\frac{\beta}{2} e^{\frac{i}{2}\phi} \\ \sqrt{2}\sin\frac{\beta}{2} e^{-\frac{i}{2}\phi} \end{pmatrix} \tag{3.152}$$

then $z = Z_1/Z_2 = \cot\frac{\beta}{2} e^{i\phi}$ which is the formula (3.89) for stereographic projection. To discover the sphere, note that $X = ZZ^\dagger$ is a Hermitian matrix with vanishing determinant in fact

$$X = \begin{pmatrix} 1+\cos\beta & \sin\beta e^{i\phi} \\ \sin\beta e^{-i\phi} & 1-\cos\beta \end{pmatrix} . \qquad \Rightarrow \qquad (t, x_1, x_2, x_3) = (1, \sin\beta\cos\phi, \sin\beta\sin\phi, \cos\beta) \,. \tag{3.153}$$

Thus the *celestial sphere* is a constant time slice of the future light cone of the origin of Minkowski spacetime. Suppose you see three stars coming towards you in three given directions. By an appropriate Lorentz transformation, you can always pass to a frame of reference in which one is due north, one is due south and one is due east.

3.14 Use of Conformal mappings to solve Laplace's equation

Suppose we wish to find a real valued solution Ψ of the *Dirichlet problem* in some complicated domain $D \subset \mathbb{C}$ in the z-plane

$$\nabla^2 \Psi = 4\frac{\partial^2 \Psi}{\partial z \partial z} = 0 \,, \qquad \text{in}\, D \,, \qquad \Psi|_{\partial D} = \Psi_o \tag{3.154}$$

We try to find a holomorphic map $w = f(z)$ which takes taking D to a simpler domain $\tilde{D} = f(D)$ in the w-plane, in which we can easily find a real valued solution Φ of the *Dirichlet problem* with the same boundary values at corresponding points of the boundary

$$\nabla^2 \Phi = 4\frac{\partial^2 \Phi}{\partial w \partial w} = 0 \,, \qquad \text{in}\, f(D) \,, \qquad \Phi|_{\partial f(D)} = \Psi_o \tag{3.155}$$

Then we know that

$$\Phi = \Re\, g(w) \tag{3.156}$$

for some holomorphic function $g(w)$ in $\tilde{D} = f(D)$. Now $h = g \circ f = g(f(z)$ is a holomorhic function in D. and thus

$$\Psi = \Re\, h(z) \tag{3.157}$$

is a certainly harmonic in D and by (3.155) it also satisfies the boundary condition (3.154) on ∂D .

Example

Let $D : y > 0,\ xy < 1,\ x > y,\ x^2 - y^2 < 1$ That is D is bounded by the real axis,on which $\Phi = 0$, the line through the origin at $45°$, on which $\Psi = 0$, and the rectangular hyperbola $xy = 1$ on which $\Psi = 1$ and the rectangular hyperbola $x^2 - y^2 = 1$ on which $\Psi = 0$. The map $w = z^2$ takes D to the rectangle $0 < u < 1$ and $0 < v < 1$. One has $\Phi = 0$, on three sides and $\Phi(u, 1) = 1$ on the top.

Separation of variables and Fourier's theorem gives

$$\Phi(u, v) = \sum b_n \sin(n\pi v) \sinh(n\pi u) \tag{3.158}$$

with

$$b_n = 0 \qquad n\,\text{even}\,, \qquad b_n = \frac{4}{n\pi \sinh(n\pi u)} \qquad n\,\text{odd} \tag{3.159}$$

Thus

$$\Phi(u, v) = \Re\Big(\sum_{n\,\text{odd}} \frac{4i\cos(w)}{n\pi \sinh(n\pi)}\Big) \tag{3.160}$$

$$\Psi(x, y) = \Re\Big(\sum_{n\,\text{odd}} \frac{4i\cos(z^2)}{n\pi \sinh(n\pi)}\Big) \tag{3.161}$$

4 Cauchy's Theorem and Contour Integrals

If $f(z)$ is complex valued function in some domain D without singularities or branch points and $\gamma(t)$ a smooth curve lying in D, so that $z(t) = \gamma(t)$, with initial point $z_i = \gamma(t_i)$ and final point $z_f = \gamma(t_f)$ then we have the

Definition

$$\int_\gamma f(z)\,dz = \int_{t_i}^{t_f} f(z(t))\frac{dz}{dt}dt \tag{4.1}$$

$$= \int_{t_i}^{t_f} (u + iv)(dx + idy) \tag{4.2}$$

$$= \int_{t_i}^{t_f} (udx - vdy) + i\int_{t_i}^{t_f} (vdx + udy)\,. \tag{4.3}$$

In this context the curve $\gamma(t)$ along which one integrates is often referred as a $contour$, even though it may not arise as a contour or level set of any particular real valued function in the problem. This is because, strictly speaking, one should distinguish between a *curve* and the corresponding *path*. A *curve* is usually defined to include its parametrization. A curve is thus a map from $\mathbb{R} \ni t \to \mathbb{R}^2 \equiv \mathbb{C} \ni z(t) = x(t) + iy(y)$ or if it is closed curve, $S^1 \ni t \to \mathbb{R}^2 \equiv \mathbb{C} \ni x + iy$ Changing the parameter changes the curve and the speed $\frac{dz}{dt}$ at which it is executed but the path or contour, i.e the point set constituting image of the map is unchanged. Since the integral (4.3) depends only on the path and not any particular parametrization, it is called a *contour integral*. We refer to a *closed contour* and a *simple closed contour* if there is some parametrization for which is is closed or simple and closed. Similarly we can chose an orientation for a contour by picking a parametrization t for it and then specifying that "forward "is in the direction of increasing t. In what follows, I shall, not distinguish very carefully between curve and contour unless confusion may arise.

Now suppose that γ is any simple closed lying in D then we have

Cauchy's Theorem states that if $f(z)$ is holomorphic in D then

$$\boxed{\oint_\gamma f(z)\,dz = 0\,.} \tag{4.4}$$

To prove this we apply *Stokes's Theorem* to the sub-domain $\hat{D} \subset D$ with boundary $\partial\hat{D} = \gamma$.

We have

$$\oint_\gamma f(z)\,dz = \oint_\gamma \mathbf{A}.\mathbf{dx} + i \oint \mathbf{B}.d\mathbf{x} \tag{4.5}$$

with

$$\mathbf{A} = (u, -v, 0)\,, \qquad \mathbf{B} = (u, v, 0)\,. \tag{4.6}$$

The CR equations (3.25) imply that

$$\operatorname{curl} \mathbf{A} = 0 = \operatorname{curl} \mathbf{B}\,. \tag{4.7}$$

There is a converse result, which we won't prove:

Morera's Theorem

if f is continuous in D and (4.4) holds $\forall\, \gamma \in D$, then $f(z)$ is holomorphic in D

4.1 Some consequences of Cauchy's Theorem

Cor.1 *Path independence*

If $f(z)$ is holomorphic in D , and D is simply connected, then

$$\int_\gamma f(z)\,dz = \int_{\gamma'} f(z)\,dz \tag{4.8}$$

for all curves γ and γ' lying in D which join the same initial and final point.

Cor.2 *Deformation Property*

Suppose that γ and γ' are any two closed curves which can be continuously deformed into one another while lying entirely within D, then

$$\oint_\gamma f(z)\,dz = \oint_{\gamma'} f(z)\,dz \tag{4.9}$$

Note that D need not necessarily be simply connected. One sometimes says that the two closed curves γ and γ' are *homotopic within* D.

Cor.3 *Anti-derivation property*

Suppose $f(z)$ is analytic in some simply connected domain D, then there exists an analytic function $F(z)$, unique up to an integration constant such that for all curves $\gamma \in D$ connecting z_i to z_f

$$\int_{z_i}^{z_f} dz = F(z_f) - F(z_i)\,. \tag{4.10}$$

This follows easily from Stokes's theorem and the CR equations. $F(z)$ is defined only up to an integration constant since if

$$F(z) = \int_{z_0}^{z} f(z')\, dz' \,, \tag{4.11}$$

where z_0 is some arbitrarily chosen point in D then (4.10) will hold for all curves in D connecting z_0 to z.

4.2 Taylor Series and singularities

One has the following, result which we shall not prove

Taylor's Theorem

If $f(z)$ is complex differentiable in a disc $D(z_0, R)$ then uniformly in $D(z_0, R)$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \,, \qquad \forall\, |z - z_0| < R} \tag{4.12}$$

In other words the series is uniformly and absolutely convergent within $D(z_0, R)$ and may be differentiated or integrated term by term. Of course

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{n!} \left| \frac{df}{dz^n} \right|_{z=z_0} \tag{4.13}$$

.

Thus *holomorphicity* is equivalent to *analyticity* in the sense of having a convergent Taylor series (which is sometimes taken as the definition of analytic). Moreover if $f(z)$ is once complex differentiable it is analytic and hence complex differentiable arbitrary many times. For that reason we have not been very careful about the existence of continuous second derivatives when discussing the harmonicity of holomorphic functions.

It is clear from Taylor's Theorem, that z_0 is not a branch point or a singular point of the function $f(x)$. Given a holomorphic function in a disc $D(z_0, R)$ one may examine the convergence of (4.12) if one extends the radius to a larger value $R' > R$. One finds that the series remains convergent as long as $D(z_0, R)$ contains no singularities. In other words, the series diverges at the singularity nearest to z_0.

4.3 Taylor-Laurent Expansions

Definition

The *open annulus* $A(z_0, a, b)$ centred on z_0 is

$$A(z_0, a, b) \;=\; \{z | a < |z - z_0| < b\}. \tag{4.14}$$

Now suppose that $f(z)$ is single valued and analytic in an annulus $A(z_0, a, b)$, then

Proposition There exists a unique expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n(z-z_0)^n \tag{4.15}$$

$$= \sum_{m=0}^{\infty} a_m(z-z_0)^m + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m} \tag{4.16}$$

Again, we shall not prove this but note that the convergence is such that one may differentiate and integrate term by term

Example

$f(z) = \frac{e^{2z}}{(z-1)^3}$, $z = z_0$, We set $u = z - 1$ so that $z = u + 1$ and

$$f = e^2 \frac{e^{2u}}{u^3} = \frac{e^2}{u^3}\big(1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{} + \dots\big) \tag{4.17}$$

$$= e^2\big(\frac{1}{u^3} + \frac{2}{u^2} + \frac{2}{u} + \frac{4}{3} + \frac{2u}{3} + \dots\big) \tag{4.18}$$

$$= e^2\big(\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2(z-1)}{3} + \dots\big) \tag{4.19}$$

The first three terms are the new ones and we see that n in (4.16) only goes down to $n = -3$.

Example

Laurent Series and Separation of Variables of Laplace's' equation in polar coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0. \tag{4.20}$$

Separation of variables leads to

$$\phi = \sum_{n=0} A_n r^n \cos(n\theta + \alpha_n) + \sum_{n=1} \frac{1}{r^n} B_n \cos(n\theta + \beta_n) + B_0 \log r \tag{4.21}$$

The two series may be expressed in terms of the real parts of a Taylor-Laurent series with $z_0 = 0$, the terms involving negative powers of r (the Laurent terms) being associated with sources at small radius. The terms with positive powers of r (the Taylor terms) are associated with sources at infinity. The $\log r$ term, which is associated with a source at the origin cannot be obtained from a Taylor-Laurent series since $\log z$ is not single valued in any annulus about the origin.

Example $f(z) = \frac{1}{z}$.

For $|z| < 1$ we have a convergent Taylor series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \tag{4.22}$$

which has a singularity at $z = 1$, on the *circle of convergence*. To obtain a representation convergent outside this circle, i.e. in the annulus $A(0, 1, \infty)$ we use the fact that

$$\begin{aligned} \frac{1}{1-z} &= -\frac{1}{z}\frac{1}{1-\frac{1}{z}} && (4.23)\\ &= -\frac{1}{z}\sum_{n=0}\frac{1}{z^n} && (4.24)\\ &= \sum_{-\infty}^{n=-1} -z^n && (4.25) \end{aligned}$$

Example $f(z) = \frac{1}{\sin z}$. Using the Taylor series for $sin(z)$ we can obtain a seres of the form

$$\begin{aligned} \frac{1}{\sin z} &= \frac{1}{z}\frac{1}{1-\frac{z^2}{6}+\frac{z^4}{120}+\dots} && (4.26)\\ &= \frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+\dots && (4.27) \end{aligned}$$

The function $g(z) = \frac{1}{\sin z} - \frac{1}{z}$ is differentiable at the origin and the series

$$g(z) = \frac{1}{\sin z} - \frac{1}{z} = \frac{z}{6} + \frac{7z^3}{360} + \dots \tag{4.28}$$

converges in the disc $D(0, \pi)$. To cancel the singularities at $z = \pm\pi$ coming from the zeros of $\sin z$ at $z = \pm\pi$ we consider

$$h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{1}{z-\pi} + \frac{1}{z+\pi} \tag{4.29}$$

which has a Taylor series in the annulus $A(0, \pi, 2\pi)$. Now using the Taylor-Laurent series obtained in the previous example one deduce that in the annulus $A(0, \pi, 2\pi)$ we have the Taylor-Laurent series

$$\frac{1}{\sin z} = (\frac{z}{6} + \frac{72z^3}{360} + \dots) - \frac{2}{\pi}\sum_{n\,\mathrm{odd}}(\frac{z}{\pi})^n + \frac{1}{z} - \frac{2}{\pi}\sum_{n\,\mathrm{even}}(\frac{\pi}{z})^n \tag{4.30}$$

4.4 Classification of singularities

Definition We say that $f(z$ has an *isolated singularity* at $z = z_0$ if the inner radius a of the annulus in the Taylor-Laurent annulus series (4.16), can be made arbitrarily small. That is (4.16) holds holds in $A(z_0, 0, b)$ for some $b > 0$.

Definition If further the the coefficients c_n vanish for $n < -N$ we say that $f(z)$ has a *pole of order* N at $z = z_0$.

Definition If $N = 1$ we say that $f(z)$ has a *simple pole* at $z = z_0$.

Definition If $N = \infty$ we say that $f(z)$ has an *essential singularity* at $z = z_0$.

Example The basic example of an essential singularity is $f(z) = e^{\frac{1}{z}}$.

Example If $f(z)$ has a branch point at $z = z_0$, then $f(z)$ has a *non-isolated* singularity at $z = z_0$.

4.5 Cauchy's Integral Formula

Suppose $f(z)$ has a Taylor-Laurent series (4.16) about $z = z_0$, then

$$\boxed{c_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz} \tag{4.31}$$

For all simple closed curves γ in the annulus $A(z_0, a, b)$

This follows immediately from the identity

$$\oint_\gamma (z-z_0)^{m+n-1}\, dz = 2\pi \delta_{m+n,0} \tag{4.32}$$

which in turns follows by setting $z - z_0 = \rho e^{i\phi}$, so that $dz = d\rho e^{i\phi} + \rho e^{i\phi} i d\phi$, integrating around a circle $\rho = \mathrm{constant}$ lying inside the annulus $A(z_0, a, b)$. The answer then follows for other curves by the Deformation Property. Alternatively if $m + n \neq 0$ we have

$$\oint_\gamma (z-z_0)^{m+n-1}\, dz = \oint_\gamma d\Big(\frac{z-z_0}{m+n}\Big) = 0\,. \tag{4.33}$$

If $m + n = 0$ we must integrate

$$\oint_\gamma \frac{dz}{z} = \oint_\gamma d(\log z) = [\,\log z\,] = 2\pi i\,. \tag{4.34}$$

Example The Bernoulli numbers B_n arise in many combinatorial problems. They are defined by

$$\frac{z}{e^z - 1} = 1 = \frac{1}{2} z + \frac{1}{12} z^2 - \frac{1}{70} z^4 - \ldots \tag{4.35}$$

$$= \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \tag{4.36}$$

One has

$$B_n = \frac{n!}{2\pi i} \oint_\gamma \frac{1}{z^n (e^z - 1)}\, dz \tag{4.37}$$

Definition The coefficient c_{-1} given a special name. It is called the *residue of the function* $f(z)$ *at* $z = z_0$

Proposition The residue c_1 of of $f(z)$ is given by

$$c_{-1} = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz \tag{4.38}$$

Read the other way: to evaluate an integral it suffices to evaluate a residue.

Cauchy's Integral Formula If $f(z)$ is analytic in a disc $D(z_0, R) = A(z_o, 0, R) \sqcup z_0$ about z_0, then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz} \tag{4.39}$$

Example

The left hand side of the Cauchy Integral Formula (4.39) is a complex valued solution of Laplace's equation (3.66) inside the curve γ. The left hand side gives it in terms of its boundary values on γ, in which $\frac{1}{z-z_0}$ plays the rôle of a type of Green's function.

Example Gauss's mean value theorem for harmonic functions.

Take $\gamma = C$ to be a circle of radius ρ centred on $z = z_0$. Then

$$f(z_0) = \oint_C f(z_0 + \rho e^{i\phi}) \frac{d\phi}{2\pi} \tag{4.40}$$

The real part of of this expression

$$u(x_0, y_0) = \oint_C u(x_0 + \rho \cos\phi, y_0 + \rho + \rho \sin\phi) \frac{d\phi}{2\pi} \tag{4.41}$$

gives the value of the harmonic function $u(x_0, y_0) = \Re f(z_0)$ at a point p in terms of its average value value around a circle enclosing p. This is a general property of harmonic functions which may also be proved using separation of variables (4.21).

Gauss' mean value theorem evidently implies that two harmonic functions with identical boundary values are themselves identical.

4.6 Consequences of Cauchy's formula

Cor.1 Cauchy's Inequality

Suppose that $|f(z)|_C < M$ on some circle of radius ρ about $z = z_0$, then

$$f^n(z_0) < \frac{n!M}{\rho^n}\,. \tag{4.42}$$

We have

$$\begin{aligned} f^n(z_0) &= \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz & (4.43)\\ &= \frac{n!}{2\pi i} \frac{1}{\rho^n} \int_0^{2\pi} e^{-in\phi} f(z_0 + \rho e^{i\phi})\, d\phi & (4.44) \end{aligned}$$

Using $|\int h(z) d\theta| \le \int |h(z)| d\theta$ we obtain

$$\begin{aligned} |f^n(z_0)| &\le \frac{n!}{\rho^n} \int_0^{2\pi} |f(z_0 + \rho e^{i\phi})| \frac{d\phi}{2\pi} & (4.45)\\ &\le \frac{n!}{\rho^n} \int_0^{2\pi} M \frac{d\phi}{2\pi} & (4.46)\\ &= \frac{n!M}{\rho^n} & (4.47) \end{aligned}$$

Cor.2 Liouville's Theorem States that if $f(z)$ is entire and bounded, $|f(z)| < M \,\forall z \in \mathbb{C}$ then $f(z)$ is constant.

From the previous result

$$|f'(z_0)| < \frac{M}{\rho}\,\forall\rho > 0 \tag{4.48}$$

and hence $f' = 0$ for all z_0 .

As a special case we deduce that a bounded harmonic function defined in all of $\mathbb{R}^2$ must be constant. In fact this last result and Gauss's Mean Value theorem hold for harmonic functions in $\mathbb{R}^n$ for all $n \geq 2$.

Cor.3 The fundamental Theorem of Algebra

Every polynomial $P(z)$ of degree n at at least 1 has at least one root. (and hence n roots).

If $P(z)$ is never zero then $f(z) = \frac{1}{P(z)}$ is analytic since $(\frac{1}{P(z)})' = -\frac{P(z)'}{P^2(z)}$. Now $|\frac{1}{P(z)}|$ is bounded as $|z| \to \infty$. Thus by Liouville $P(z)$ is constant which is a contradiction.

5 Residue Calculus

The basic result is

Proposition

Suppose that $f(z)$ is analytic inside a simple closed curve γ with the exception of a finite number n of isolated singularities $z = z_k$ at which the residue is $\mathrm{Res}[f, z_k]$. then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{k=n} \mathrm{Res}[f, z_k]} \tag{5.1}$$

To prove this one deforms γ into n small contours γ_k such that γ_k encloses only the k'th residue to together with k non-intersecting "railway contours "connecting γ_k and γ.

Example Counting zeros of Polynomials

Suppose $P(z) = a_0 + a_1 z + \ldots a_n z^n$, $a_n \neq 0$ is a polynomial of degree n. Then $f(z) = \frac{P'(z)}{P(z)}$ is analytic away from the zeros of $P(z)$. Near an isolated simple root at $z = z_k$

$$P(z) = (z - z_k) g_k(z) \tag{5.2}$$

where $g_k(z)$ is analytic and $g_k(z_k) \neq 0 \neq g'(z_k))$. (In fact $g_k(z)$ is a a polynomial of degree $(n-1)$) Thus

$$\frac{P'(z)}{P(z)} = \frac{g_k(z) + (z - z_k) g_k'(z)}{(z - z_k) g_k(z)} \tag{5.3}$$

$$= \frac{1}{z - z_k} + \frac{g_k'(z)}{g_k(z)} \tag{5.4}$$

has residue 1. Near a root of multiplicity n_k we set $P(z) = (z - z_k)^{n_k} h_k(z)$ with $h(z_k) \neq 0$ and find that

$$\frac{P'(z)}{P(z)} = \frac{n_k}{z - z_k} + \frac{h'(z)}{h(z)} \tag{5.5}$$

The residue is therefore $\mathrm{Res}[\frac{P'}{P}, z_k] = n_k$ We integrate $\frac{P'}{P}$ around a large circle at infinity and use the fact that at large z

$$\frac{P'(z)}{P(z)} \approx \frac{n}{z} \tag{5.6}$$

Evaluating the integral we get

$$n = \sum n_k \tag{5.7}$$

In other words the sum of the zeros counted with respect to multiplicity equals the degree of the polynomial.

Definition A holomorphic function is said to be *meromorphic* if its only singularities are isolated poles.

In fact such a function may be expressed as the ratio of two entire functions.

Suppose $f(z)$ is has a pole of order m_i at $z = z_i$,then near $z = z_i$ we have

$$f = \frac{1}{(z - z_i)^{m_i}} g_i(z) \tag{5.8}$$

Then

$$\frac{f'}{f} = -\frac{m_i}{(z - z_i)} + \frac{g'(z)}{g_i(z)} \tag{5.9}$$

The residue $\mathrm{Res}[\frac{f'}{f}, z_i] = -m_i$.

Now if $f(z)$ is holomorphic inside γ except for a finite number of isolated poles and zeros, and non-vanishing on γ then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{\text{zeros}} n_k - \sum_{poles} m_i \,. \tag{5.10}$$

Example $f = e^{-\frac{2}{z}} = 1 - \frac{2}{z} + \frac{1}{2}(\frac{2}{z})^2 + \dots$ has a pole of infinite order at the origin with residue $\mathrm{Res}[e^{-\frac{2}{z}}, 0] = -2$. Thus for simple contours γ enclosing the origin

$$\oint_\gamma e^{-\frac{2}{z}}\, dz = -4\pi i \tag{5.11}$$

Choosing for γ a circle centred of radius a and take real and imaginary parts, we obtain

$$\int_0^{2\pi} d\theta\, e^{-\frac{2\cos\theta}{a}} \cos(\theta + \frac{2\sin\theta}{a}) \quad = \quad -\frac{4\pi}{a} \tag{5.12}$$

$$\int_0^{2\pi} d\theta\, e^{-\frac{2\cos\theta}{a}} \sin(\theta + \frac{2\sin\theta}{a}) \quad = \quad 0\,. \tag{5.13}$$

The second integral is trivially zero, but the first is not so obvious.

Example *Show that* $\int_0^{2\pi} \cos^{2n}\theta\, d\theta = 2\pi \frac{(2n)!}{4^n (n!)^2}$.

If γ is the unit circle, one has

$$\frac{1}{4^n} \oint_\gamma \big(z + \frac{1}{z}\big)^{2n} \frac{dz}{z} = \int_0^{2\pi} \cos^{2n}\theta\, d\theta\,. \tag{5.14}$$

On the other hand one may use the Binomial theorem to obtain a Taylor-Laurent series about the origin. and evalute the residue of the simple pole at the origin.

Example Evaluate $I = \int_0^{2\pi} h(\cos\theta, \sin\theta)\, d\theta$

We substitute $e^{i\theta} = \frac{z}{a}$

$$\cos\theta = \frac{1}{2}(\frac{z}{a} + \frac{a}{z})\,, \qquad \sin\theta = \frac{1}{2i}(\frac{z}{a} - \frac{a}{z}) \tag{5.15}$$

Thus

$$I = \int_0^{2\pi} h(\frac{1}{2}(\frac{z}{a} + \frac{a}{z}), \frac{1}{2i}(\frac{z}{a} - \frac{a}{z}))d\theta \tag{5.16}$$

$$= \int_\gamma h(z)\frac{dz}{iz}\,, \tag{5.17}$$

where $h(z) = h(\frac{1}{2}(\frac{z}{a} + \frac{a}{z}), \frac{1}{2i}(\frac{z}{a} - \frac{a}{z}))$ and the contour γ is initially a circle of radius a centred on the origin. One now uses Cauchy's theorem to evaluate the integral. This requires efficient methods for

5.1 Calculating Residues

Example If $f(z)$ has a simple pole at $z = z_0$ then

$$f(z) = c_{-1}\frac{1}{(z - z_0)} + c_0 + c_1(z - z_o) + \dots \tag{5.18}$$

and

$$\boxed{c_{-1} = \lim_{z\to z_0}(z - z_0)f(z)} \tag{5.19}$$

Example If $f(z)$ has a pole of order n at $z = z_0$ then

$$f(z) = \frac{c_{-n}}{(z - z_o)^n} + \frac{c_{1-n}}{(z - z_o)^{n-1}} + \dots \frac{c_{-1}}{(z - z_0)} + c_0 + c_1(z - z_0) + \dots \tag{5.20}$$

and

$$(z - z_0)^n f(z) = c_{-n} + c_{1-n}(z - z_o) + \dots c_{-1}(z - z_o)^{n-1} + \dots \tag{5.21}$$

Thus

$$\boxed{\frac{d^{n-1}}{dz^{n-1}}(z - z_o)^n f(z)\Big|_{z=z_0} = (n-1)!\; c_{-1}\,.} \tag{5.22}$$

Example $f = \frac{z}{(z+1)^2(z-1)}$ has a simple pole with residues $\frac{1}{4}$ at $z = 1$ and a pole of order 2 at $z = -1$ with residue $-\frac{1}{4}$.

Example Evaluate

$$\int_0^{2\pi} \frac{d\theta}{(1 + 3\cos^2\theta)} = \oint_{|z|=1} \frac{dz}{iz}\frac{1}{1 + \frac{3}{4}(z + \frac{1}{z})^2} \tag{5.23}$$

$$= \oint_{|z|=1} dz\frac{-4iz}{3z^4 + 10z^2 + 3} \tag{5.24}$$

$3z^4 + 10z^2 + 3$ has four simple roots, two outside the unit circle at $z = \pm i\sqrt{3}$ and two inside the unit circle $z = \pm\frac{i}{\sqrt{3}}$. They give four simple poles and we need the residues of the latter two. Both of these residues are found using the limit method to be $-\frac{1}{4}$ and hence

$$\int_0^{2\pi} \frac{d\theta}{(1 + 3\cos^2\theta)} = \pi\,. \tag{5.25}$$

Example $\int_\gamma \frac{1}{z^3-1}\,dz$, with (i) $\gamma = \{|z| = 1\}$ and (ii) $\gamma = \{|z-1| = 1\}$. There are simple poles at the cube roots of unity $z = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ with residues $\frac{1}{3}, \frac{e^{\frac{-4\pi i}{3}}}{3}, \frac{e^{\frac{-2\pi i}{3}}}{3}$ respectively. In case (i) all three lie inside the unit circle and the integral is $\frac{2\pi i}{3}(1 + e^{\frac{-4\pi i}{3}} + e^{\frac{-2\pi i}{3}}) = 0$. In case (ii) only $z = 1$ lies inside the contour and the integral equals $\frac{2\pi}{3}i$.

5.2 Integrating around branch cuts

Example Find $I = \oint_\gamma (z^2-1)^{\frac{1}{2}}dz$ where we slit the plane along the real axis between the branch points at $z = \pm 1$ an chose the branch of the function which is real and positive on the real axis to the right of the branch point at $z = 1$ and the contour is taken to enclose the branch cut.

$$(z^2-1)^{\frac{1}{2}} = z - \frac{1}{2z} + \ldots \implies \mathrm{Res}[(z^2-1)^{\frac{1}{2}}, 0] = -\frac{1}{2} \implies I = -\pi i\,. \tag{5.26}$$

We can check this by shrinking the contour onto the cut and taking into account the fact that $(z^2-1)^{\frac{1}{2}}$ suffers a discontinuity across the cut . Temporarily ignoring the contribution from the branch points at $z = \pm 1$ we have in the limit

$$I = \int_{+1}^{-1} i\sqrt{1-x^2}\,dx - \int_{-1}^{+1} i\sqrt{1-x^2}\,dx = -2i\int_{-1}^{+1}\sqrt{1-x^2}\,dx = -i\pi \tag{5.27}$$

The contribution from branch points can be estimated by considering the contribution from a sector of a small semi-circle of radius ϵ centred on ± 1. This is

$$I_\epsilon^\pm = \int_{\mp\frac{\pi}{2}}^{\pm\frac{\pi}{2}} \epsilon i e^{i\theta}(z^2-1)^{\frac{1}{2}}\,d\theta \tag{5.28}$$

Since $|(z^2-1)^{\frac{1}{2}}| < \epsilon^{\frac{1}{2}}M$ for some constant M

$$\left|I_\epsilon^\pm\right| < \pi\epsilon^{\frac{3}{2}}M \implies \lim_{\epsilon\downarrow 0} I_\epsilon^\pm = 0\,. \tag{5.29}$$

5.3 Integrals along the real axis or positive real axis

Proposition Suppose $f(x)$ is such that

- (i) $f(x)$ is the real part of function $f(z)$ which is meromorphic function in the upper half plane. That is has a finite number of isolated poles $z = z_i \in UHP$.

- (ii) $\lim_{|z|\uparrow\infty} zf(z) = 0$ in the UHP.

Then

$$I = \int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{z_i \in UHP} \mathrm{Res}[f(z), z_i] \tag{5.30}$$

Example $I = \int_{-\infty}^{\infty} \frac{dx}{x^6+1}$. The meromorphic function $\frac{1}{z^6+1}$ has six isolated simple poles at $z_n = e^{\frac{(2n+1)\pi i}{6}}$, $n = 0, 1, 2, 3, 4, 5$. of which three lie in the UHP, $z_0, z_1, z_2 \in UHP$. Since $\mathrm{Res}[\frac{1}{z^6+1}, e^{\frac{(2n+1)\pi i}{6}}] = \frac{1}{6}e^{-\frac{15(2n+1)\pi i}{6}}$, one finds that $I = \frac{2\pi}{3}$.

Example $I = \int_0^{\infty} \frac{dx}{x^3+1}$. Since x^3 is odd, we cannot extend the integral to $-\infty < x < \infty$ and take half the result. However z^3 is invariant under rotations through 120° . Thus we are inspired to consider $J = \int_\gamma \frac{1}{z^3+1}\,dz$, where γ starts from 0 and proceeds along the real axis to $x = a$. We then take γ to proceed 120° along a circle of radius a in the UHP; γ then returns to 0 along the radial line $\theta = \frac{2\pi}{3}$. The contour so defined encloses just one simple pole at $z = e^{\frac{i\pi}{3}}$ with residue $\mathrm{Res}[\frac{1}{z^3+1}, \frac{i\pi}{3}] = \frac{1}{3}e^{-\frac{2\pi i}{3}}$. Taking the limit $a \uparrow \infty$ we drop the integral over the arc of the circle and obtain

$$J = \frac{2\pi i}{3}e^{-\frac{2\pi i}{3}} = \int_0^{\infty} \frac{dx}{x^3+1} + \int_{\infty}^{0} \frac{e^{\frac{2\pi i}{3}}\,dr}{r^3+1} = (1 - e^{\frac{2\pi i}{3}})I \implies I = \frac{2\pi}{\sqrt{27}}\,. \tag{5.31}$$

5.4 The Keyhole Contour

An alternative method for evaluating the previous integral is to consider the integral

$$K = \int_\gamma \frac{\log z}{z^3+1}\,dz \tag{5.32}$$

where the function $\log z$ is taken to have a branch cut along the positive real axis and to take the value $\log x$ just above and $\log x + 2\pi i$ just below it. The contour γ starts from 0 and along the real axis just above the cut out to $x = a$. It follows a full circle of radius a returning to the real axis just below the cut. The contour then returns to zero along the real axis just below the cut. For large a, there are three simple poles inside γ at $z = \{z_1, z_2, z_3\} = \{e^{\frac{\pi i}{3}}, e^{\pi i}, e^{\frac{5\pi i}{3}}\}$ with residues $\frac{\log z_k}{3z_k^2}$

In the limit $a \uparrow \infty$

$$K = \int_0^{\infty} \frac{\log x}{x^3+1}\,dx + \int_{\infty}^{0} \frac{\log x + 2\pi i}{x^3+1}\,dx = -2\pi i I = \sum_z 2\pi i \frac{\log z_k}{3z_k^2} = -2\pi i\,(\frac{2\pi}{\sqrt{27}}) \tag{5.33}$$

Example

$$I = \int_0^{\infty} \frac{x^{m-1}}{1+x^2}\,dx\,, \qquad 0 < m < 2\,; \quad m \neq 1 \tag{5.34}$$

Let

$$J = \oint_\gamma \frac{z^{m-1}}{1+z^2}\,dz\,, \tag{5.35}$$

where γ is the keyhole contour defined above. The integrand has a branch point at $z =$ and simple poles at $z = \pm i$. with residues $-\frac{1}{2}e^{\frac{(2\pm 1)m\pi i}{2}}$ respectively. As long as $m < 2$, we may ignore the contribution from the large semi circle at infinity and so

$$J = (1 - e^{2m\pi i}) \int_0^\infty \frac{x^{m-1}}{1+x^2}\, dx \implies I = \pi \cot(\frac{m\pi}{2})\,. \tag{5.36}$$

5.5 Jordan's Lemma

We have seen that if $m > 2$ the contour around a semi circle $\gamma = \cap_a$ in the upper half plane of radius a cannot be neglected in the limit $a \uparrow \infty$. However we do have

Jordan's Lemma

$$\boxed{|zf(z)| < M \text{as} \, |z| \to \infty \implies \lim_{a\uparrow\infty} \int_{\cap_a} e^{imz} f(z)\, dz = 0\,, \quad m > 0} \tag{5.37}$$

$$\left| \int_{\cap_a} e^{imz} f(z)\, dz \right| \le M \int_0^\pi e^{-ma\sin\theta}\, d\theta = 2M \int_0^{\frac{\pi}{2}} e^{-ma\sin\theta}\, d\theta \tag{5.38}$$

But for $0 \le \theta \le \frac{\pi}{2}$

$$\sin\theta \ge \frac{2\theta}{\pi} \implies 2M \int_0^{\frac{\pi}{2}} e^{-ma\sin\theta}\, d\theta \le 2M \int_0^{\frac{\pi}{2}} e^{-2ma\frac{\theta}{\pi}}\, d\theta = M\frac{\pi}{ma}(1 - e^{-ma})\,, \tag{5.39}$$

whence the result follows.

Example

$$I = \int_{-\infty}^{\infty} \frac{\cos(mx)}{x^2+1}\, dx = \Re \oint_\gamma \frac{e^{imz}}{(z-i)(z+i)}\, dz \qquad m > 0\,, \tag{5.40}$$

the contour being a semi-circle of radius a in the UHP. The pole at $z = i$ has residue $\frac{e^{-m}}{2i}$ and hence

$$I = \pi e^{-m}\,. \tag{5.41}$$

5.6 Using contour integrals to obtain Cauchy's Principal Value

Definition

Given a real function $f(x)$ defined on some interval on $[x_{-1}, x_0) \sqcup (x_0, x_{+1})]$, then *Cauchy's Principal Value* of its integral from x_{-1} to x_{+1} is, if the limit exists,

$$\boxed{PV \int_{x_{-1}}^{x_{+1}} f(x)\, dx = \lim_{\epsilon\downarrow 0} \quad \Big(\int_{x_{-1}}^{-\epsilon} f(x)\, dx + \int_{\epsilon}^{x_{+1}} f(x)\, dx \Big)} \tag{5.42}$$

Example

$$PV \int_{-1}^{!} \frac{dx}{x^3} = \lim_{\epsilon\downarrow 0} \quad \Big([-\frac{1}{2x^2}]_{-1}^{\epsilon} + [-\frac{1}{2x^2}]_{\epsilon}^{1} \Big) = 0\,. \tag{5.43}$$

It is sometimes possible to evaluate Cauchy's Principal Value using an appropriately chosen contour integral.

Example

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{\sin\ x}{x}\, dx && (5.44)\\
&= \lim_{\epsilon\downarrow 0}\Big(\int_{-\infty}^{-\epsilon} \frac{sin\, x}{x}\, dx + \int_{\epsilon}^{\infty} \frac{sin\, x}{x}\, dx\Big) && (5.45)\\
&= \lim_{\epsilon\downarrow 0}\Big(\int_{-\infty}^{-\epsilon} \frac{e^{ix}}{ix}\, dx + \int_{\epsilon}^{\infty} \frac{e^{ix}}{ix}\, dx\Big) && (5.46)
\end{aligned}
$$

Now

$$J = \oint_{\gamma} \frac{e^{iz}}{z}\, dz = 0, \qquad (5.47)$$

where the contour $\gamma = CDABC = \gamma_1 \sqcup \gamma_2 \sqcup \gamma_3 \sqcup \gamma_4$ is

$$\{\epsilon < x < a; y = 0\} \sqcup \{UHP \supset |z| = a\} \sqcup \{-a < x < -\epsilon; y = 0\} \sqcup \{UHP \supset |z| = \epsilon\} \qquad (5.48)$$

Thus $CD = \gamma_1$ runs along the real axis from $x = \epsilon$ to $x = a$ and γ_4 runs $180°$ clockwise around the semi-circle $|z| = \epsilon; y > 0$. Now

$$J = \int_A^B \frac{e^{iz}}{z}\, dz + \int_B^C \frac{e^{iz}}{z}\, dz + \int_C^D \frac{e^{iz}}{z}\, dz + \int_D^A \frac{e^{iz}}{z}\, dz \qquad (5.49)$$

and

$$
\begin{aligned}
\lim_{a\uparrow\infty} \quad \int_D^A \frac{e^{iz}}{z}\, dz &= 0 && (5.50)\\
\lim_{a\uparrow\infty} \quad \int_A^B \frac{e^{iz}}{z}\, dz + \int_C^D \frac{e^{iz}}{z}\, dz &= \int_{-\infty}^{-\epsilon} \frac{e^{ix}}{ix}\, dx + \int_{\epsilon}^{\infty} \frac{e^{ix}}{ix}\, dx\, . && (5.51)
\end{aligned}
$$

Thus

$$i \int_{\infty}^{\infty} \frac{\sin\ x}{x}\, dx = -\lim_{\epsilon\downarrow 0} \quad \int_{\pi}^{0} e^{i\epsilon(\cos\theta + i\sin\theta)}\, id\theta \qquad (5.52)$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi\, . \qquad (5.53)$$

Example *Relation Cauchy's Principal value to the delta function* If $\epsilon > 0$, and $f(x)$ has no singularities in the interval $[a, b]$ of the real axis we have, if $a < x_0 < b$

$$\int_a^b \frac{f(x)}{x - x_0 \mp i\epsilon}\, dx \ = PV \int_a^b \frac{f(x)}{x - x_0}\, dx \pm i\pi f(x_0)\, . \qquad (5.54)$$

Thus considered as distributions, i.e. under the integral sign,

$$\frac{1}{x - x_0 - i\epsilon} = PV\frac{1}{x - x_0} + i\pi\delta(x - x_0)\,, \tag{5.55}$$

$$\frac{1}{x - x_0 + i\epsilon} = PV\frac{1}{x - x_0} - i\pi\delta(x - x_0)\,. \tag{5.56}$$

Subtracting

$$2\pi i\delta(x - x_0) = \frac{1}{x - x_0 - i\epsilon} - \frac{1}{x - x_0 + i\epsilon} = \frac{2i\epsilon}{(x - x_0)^2 + \epsilon^2}\,, \tag{5.57}$$

and therefore

$$\delta(x - x_0) = \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\,\frac{\epsilon}{(x - x_0)^2 + \epsilon^2}\,. \tag{5.58}$$

5.7 Contour Integrals used to sum infinite series

The function $\cot(\pi z)$ is periodic period 1 and near zero,

$$\cot(\pi z) = \frac{1}{\pi z} - \frac{\pi z}{3} + \ldots\,, \tag{5.59}$$

therefore

$$\text{Res}[\cot(\pi z), n] = \frac{1}{\pi} \qquad n \in \mathbb{Z}\,. \tag{5.60}$$

Thus

$$\text{Res}[\frac{\cot(\pi z)}{z^2}, n] = \frac{1}{n^2\pi} \qquad n \in \mathbb{Z}\setminus 0\,. \tag{5.61}$$

$$\text{Res}[\frac{\cot(\pi z)}{z^2}, 0] = -\frac{\pi}{3}\,. \tag{5.62}$$

Hence

$$\oint_\gamma \frac{\cot(\pi z)}{z^2}\,dz = 2\pi i\Big(-\frac{\pi}{3} + 2\sum_{n=1}^{n=N}\frac{1}{\pi n^2}\Big)\,, \tag{5.63}$$

where the contour γ is a square with corners $\pm(N + \frac{1}{2})(1 \pm i)$

Now

$$\cot(\pi z) = \frac{\cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\sinh(\pi y)}{\sin(\pi x)\cosh(\pi y) - i\cos(\pi x)\sinh(\pi y)} \tag{5.64}$$

is bounded on the contour and therefore

$$\lim_{N\uparrow\infty} \quad \oint_\gamma \frac{\cot(\pi z)}{z^2}\,dz = 0\,, \tag{5.65}$$

Hence

$$\sum_{n=1}^{n=\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \tag{5.66}$$

Definition The *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{n=\infty} \frac{1}{n^s} \qquad \Re s > 1\,. \tag{5.67}$$

and so one has $\zeta(2) = \frac{\pi^2}{6}$.

6 Fourier and Laplace Transforms

6.1 The Fourier Transform

Definition In the *space domain*, The Fourier Transform, FT, of a real valued function $f(x)$ is, when the integral exists, given, as a function of *wave number* $k = \frac{2\pi}{\lambda}$, where λ is the *wavelength*, by

$$\boxed{\mathcal{F}(f(x)) = \tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx} \tag{6.1}$$

or in the *time domain* a function of *angular frequency* $\omega = 2\pi\nu$, where ν is the *frequency*

$$\mathcal{F}(f(t)) = \tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \tag{6.2}$$

From the definition we have the following properties

$$(i) \quad \text{Linearity :} \qquad \mathcal{F}(f_1 + f_2) = \mathcal{F}(f_1) + \mathcal{F}(f_1) \tag{6.3}$$

$$(ii) \quad \text{Translation :} \qquad \mathcal{F}(f(x-a)) = e^{-ika}\mathcal{F}(f(x)) \tag{6.4}$$

$$(iii) \quad \text{FrequencyShift :} \qquad \mathcal{F}(e^{ik'x} f(x)) = \tilde{f}(k - k'(\tag{6.5}$$

$$(iv) \quad \text{Scaling :} \qquad \mathcal{F}(f(ax)) = \frac{1}{a}\tilde{f}(\frac{k}{a}) \tag{6.6}$$

$$(v) \quad \text{Derivation :} \qquad \mathcal{F}\left(\frac{df}{dx}\right) = ik\tilde{f}(k) \tag{6.7}$$

$$(vi) \quad \text{Multiplication :} \qquad \mathcal{F}(xf(x)) = i\frac{d\tilde{f}}{dk}\,. \tag{6.8}$$

$$\tag{6.9}$$

Example A Gaussian

$$f(x) = e^{-\frac{x^2}{2a}} \implies a\frac{df}{dx} + xf = 0 \implies aik\tilde{f} + \frac{d\tilde{f}}{dk} = 0 \implies \tilde{f} = Ae^{-\frac{ak^2}{2}}\,. \tag{6.10}$$

But

$$\tilde{f}(0) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2a}}\, dx = \sqrt{2\pi a} \implies \tilde{f} = \sqrt{2\pi a}e^{-\frac{ak^2}{2}}\,. \tag{6.11}$$

.

6.2 Evaluation of Fourier Transform by Contour Integration and use of Jordan's Lemma

Example $f(x) = \frac{1}{x^2+a^2}$, $0 < a \in \mathbb{R}$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} \frac{dx}{a^2 + x^2} \tag{6.12}$$

For $k < 0$ we can close the contour in the UHP using a large semi-circle. The contribution from the large semi circle vanishes in the limit by Jordan's Lemma and we pick up the contribution from the pole at $x = ia$. For $k > 0$ we close the contour in the lower half plane, LHP We obtain

$$\tilde{f}(k) = \mathcal{F}(\frac{1}{x^2 + a^2}) = \frac{\pi}{a} e^{-a|k|} \, . \tag{6.13}$$

Note that in this example, while $f(x)$ has a meromorphic extension to the complex x plane with just two isolated simple poles $\tilde{f}(k)$ has a whole line of non-isolated singularities on the imaginary axis of the complex k plane

6.3 The Fourier Inversion Theorem

This states that if $f(x) \in L^1 \cap L^2$, that is

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty \qquad and \qquad \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \, , \tag{6.14}$$

then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) = \lim_{\epsilon \downarrow 0} \quad \frac{1}{2} \big(f(x+\epsilon) + f(x-\epsilon) \big) \, . \tag{6.15}$$

Example $f(x) = 0$, $x < 0$, $f(x) = e^{-ax}$ $x > 0$, $a > 0$. Thus

$$\tilde{f}(k) = \int_0^{\infty} e^{-(ikx+ax)} \, dx = \frac{1}{a + ik} \, . \tag{6.16}$$

The inverse Fourier Transform FT is

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \frac{1}{a + ik} \, dk \tag{6.17}$$

The integrand has a single isolated pole in the UHP and Jordan's Lemma applies so if $x < 0$ we complete in the LHP and obtain $f(x) = 0$ for $x < 0$. If $x > 0$ we complete in the UHP and pick up the residue contribution giving $f(x) = e^{-ax}$ for $x > 0$. If $x = 0$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a + ik} \, dk = \frac{1}{2\pi} \int_{-\infty}^{0} \frac{1}{a + ik} \, dk + \frac{1}{2\pi} \int_0^{\infty} \frac{1}{a + ik} \, dk \tag{6.18}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \left(\frac{1}{a + ik} + \frac{1}{a - ik} \right) \tag{6.19}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \, \frac{2a}{a^2 + k^2} \tag{6.20}$$

$$= \frac{1}{\pi} \Big[\arctan(\frac{k}{a}) \Big]_0^{\infty} = \frac{1}{2} \tag{6.21}$$

as expected.

6.4 The Convolution Theorem

Definition The *convolution* $f \star g = g \star f$ of two real valued functions f and g is, when it exists,

$$f \star g\,(u) = \int_{-\infty}^{\infty} f(x) g(u-x)\,dx = g \star f\,(u)\,. \tag{6.22}$$

The Convolution Theorem states that

$$\boxed{\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)} \tag{6.23}$$

6.5 Parseval's Theorem

States that

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk} \tag{6.24}$$

Example We saw before that $\mathcal{F}\big(e^{-\frac{x^2}{2a}}\big) = Ae^{-\frac{ak^2}{2}}$. Thus

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{a}}\,dx = \frac{A^2}{2\pi}\int_{-\infty}^{\infty} e^{-ak^2}\,dk \tag{6.25}$$

Now set $u = \frac{x}{\sqrt{a}}$, $v = \sqrt{a}k$ and find

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{A^2}{2\pi a}\int_{-\infty}^{\infty} e^{-v^2}\,dv \implies A = \sqrt{2\pi a}\,, \tag{6.26}$$

as before.

Definition the function of wave number $P(k) = |\tilde{f}(k)|^2$ is called the *power spectrum* of $f(x)$.

Definition The *auto-correlation function* of a real valued function $f(x)$ is

$$\begin{aligned} \rho(x) &= \int_{-\infty}^{\infty} f(x - \frac{1}{2}u) f(x + \frac{1}{2}u)\,du && (6.27)\\ &= \rho(-x) && (6.28)\\ &= \int_{-\infty}^{\infty} f(x) f(x+u)\,du && (6.29)\\ &= f(x) \star f(-x)\,. && (6.30) \end{aligned}$$

Thus

$$\mathcal{F}(\rho) = \mathcal{F}(f(x))\,\mathcal{F}(f(-x)) = \mathcal{F}(f(x))\,\overline{\mathcal{F}(f(x))} = |\mathcal{F}(f)|^2 = |\tilde{f}(k)|^2 = P(k)\,. \tag{6.31}$$

In other words *the Fourier transform of the auto correlation function equals the power spectrum*

6.6 Solving O.D.E.'s

Example A *forced damped simple harmonic oscillator*

$$\frac{d^2y}{dt^2} + \frac{2}{\tau}\frac{dy}{dt} + \omega_0^2 y = f(t) \qquad \tau > 0\,. \tag{6.32}$$

Taking the FT and solving gives

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{f}(\omega)}{\omega_0^2 - \omega^2 + \frac{2i\omega}{\tau}} \tag{6.33}$$

If we assume $f(t) = \delta(t) \Longleftrightarrow \tilde{f}(\omega) = 1$ then $y(t) = G(t)$ will be the *impulse response or Green's function* with Fourier transform .

$$\tilde{G}(\omega) = \frac{1}{\omega_0^2 - \omega^2 + \frac{2i\omega}{\tau}} \tag{6.34}$$

The integrand has poles at $\omega = \frac{i}{\tau} \pm \sqrt{\omega_0^2 - \frac{1}{\tau^2}}$ There are three cases

$$\omega_0^2 > \frac{1}{\tau^2} \qquad \text{under damped} \tag{6.35}$$

$$\omega_0^2 = \frac{1}{\tau^2} \qquad \text{critically damped} \tag{6.36}$$

$$\omega_0^2 < \frac{1}{\tau^2} \qquad \text{over damped} \tag{6.37}$$

In all three cases the poles lie in the UHP. If $t < 0$ we can complete the contour in the LHP and find $G(t)$ for $t < 0$. If $t > 0$ we can complete the contour in the UHP and obtain a solution with two damped oscillations, called transients, in the under damped case and a a non-oscillatory transient if it is critical or over damped. For a general $f(t)$

$$\tilde{y}(\omega) = \tilde{f}(\omega)\tilde{G}(\omega) \tag{6.38}$$

and by the Convolution Theorem implies that

$$\boxed{y(t) = G \star F = \int_{-\infty}^{\infty} G(\tau) f(t-\tau)\, d\tau\,.} \tag{6.39}$$

Note that the FT method picks out the so called *Causal Solution* : if the force $f(t)$ vanishes for $t < t_i$ then the solution vanishes for $t < t_i$. For that reason one refers to impulse response $G(t)$ as the *Causal* or sometimes *Retarded* Green's function.

6.7 General Linear Systems

The equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V(t) \tag{6.40}$$

governs the charge $Q(t)$ and current $I = \frac{dQ}{dt}$ flowing through an electric *LCR circuit* with *capacitance* C, *resistance* R, *inductance* L and *applied voltage* $V(t)$. Taking a FT gives

$$\Big(iL\omega + R + \frac{1}{i\omega C}\Big)\tilde{I}(\omega) = \tilde{V}(\omega) = Z(\omega)\tilde{I}(\omega)\,, \tag{6.41}$$

where

$$Z(\omega) = iL\omega + R + \frac{1}{i\omega C} \tag{6.42}$$

is called the *complex impedance* of the circuit. By combining circuits in series and parallel according to the rules

$$(i) \qquad \text{series} \qquad Z_3(\omega) = Z_1(\omega) + Z_2(\omega)\,, \tag{6.43}$$

$$(ii) \qquad \text{parallel} \qquad \frac{1}{Z_3(\omega)} = \frac{1}{Z_1(\omega)} + \frac{1}{Z_2(\omega)}\,, \tag{6.44}$$

one obtains the impedance $Z(\omega)$ of a general circuit. It is clear that in general, $Z(\omega)$ will be a *rational function* that is ratio of two polynomials

$$Z(\omega) = \frac{P_1(\omega)}{P_2(\omega)}\,. \tag{6.45}$$

and hences a meromorphic function of ω. In general $Z(\omega)$ will have a finite number of isolated simple poles.

Since

$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}V(\omega)}{Z(\omega)}\,d\omega \;=\; \int_{-\infty}^{\infty} G(\tau)V(t-\tau)\,d\tau \tag{6.46}$$

$$= \int_{-\infty}^{\infty} G(t-\tau)V(\tau)\,. \tag{6.47}$$

with $\tilde{G}(\omega) = \frac{1}{Z(\omega)}$, so that

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{Z(\omega)}\,d\omega \tag{6.48}$$

the discussion above applies as long as $\lim_{\omega\to\infty} \quad \omega Z(\omega) = 0$. For Causality and Stability to hold, $Z(\omega)$ must be holomorphic (with no poles) in the LHP. In the case of electric circuits, this will be true if the resistances, capacitance's and inductance's are all non-negative.

Example Cauchy's Integral Formula and Principal Part and Kramers and Kroenig's Dispersion Relation

If we take $\omega \in \mathbb{R}$ and run the ω' contour γ along the real axis and, passing just below the singularity at $\omega' = \omega$ we have

$$-(\pi i)\left(\Re\tilde{G}(\omega) + i\Im\tilde{G}(\omega)\right) + PV\int_{-\infty}^{\infty} \frac{\{\Re\tilde{G}(\omega') + i\Im\tilde{G}(\omega))}{(\omega'-\omega)}\,d\omega' = 0\,. \tag{6.49}$$

The first term comes from evaluating the integral around the small semi-circle *just below* [6]. Thus

$$\pi\Re\tilde{G}(\omega) = -PV\int_{-\infty}^{\infty}\frac{\Im\tilde{G}(\omega)}{(\omega'-\omega)}d\omega' \tag{6.50}$$

$$\pi\Im\tilde{G}(\omega) = PV\int_{-\infty}^{\infty}\frac{\Re\tilde{G}(\omega)}{(\omega'-\omega)}d\omega'\,. \tag{6.51}$$

The formulae (6.51) called *Kramers Kroenig* or *dispersion* relations and they tell us that there are fundamental physical limits arising from Causality on the frequency characteristics of physical devices. Thus one might try try to build a perfect filter with zero phase distortion. This would have $\Im\tilde{G}(\omega)=0$. By Kramers Kroenig it would unfortunately also have $\Re\tilde{G}(\omega)=0$.

Remark The second line of (6.47) is an *infinite dimensional matrix multiplication*. Think of $V(t)$ and $I(t)$ as belonging to an infinite dimensional vector space and (6.470 as a linear map between them. Re-writing it as

$$I(t)=\int_{-\infty}^{\infty}G(t,\tau)V(\tau)\,d\tau\,,\qquad G(t,\tau)=G(t-\tau)\,, \tag{6.52}$$

and comparing with the finite dimensional analogue

$$I_i=\sum_j G_{ij}V_j \tag{6.53}$$

we see that

$$\sum_i\leftrightarrow\int d\tau\qquad i\leftrightarrow t\qquad j\leftrightarrow\tau\qquad G_{ij}\leftrightarrow G(t,\tau)=G(t-\tau) \tag{6.54}$$

Formally, therefore

$$G(t-\tau)=\delta^{-1}(t-\tau)\,. \tag{6.55}$$

A *general linear system* is often thought of as made up of *Black boxes* or *input-output devices* each of which has its own impedance function $Z(\omega)$ and *transmittance function* $Y(\omega)=\frac{1}{Z(\omega)}$. Of course $Y(\omega)$ is just what we have called $\tilde{G}(\omega)$ above.

Placing boxes in parallel corresponds to addition of transmittance functions

$$Y_3(\omega)=Y_2(\omega)+Y_1(\omega) \tag{6.56}$$

which corresponds to addition of matrices

$$G_3(t,\tau)=G_1(t,\tau)+G_2(t,\tau) \tag{6.57}$$

Placing boxes in series corresponds to multiplication of transmittance functions

$$Y_3(\omega)=Y_2(\omega)Y_1(\omega) \tag{6.58}$$

which, by the Convolution Theorem is a form of matrix multiplication

$$G_3(t,\tau)=\int_{-\infty}^{\infty}G_2(t,\tau')G_1(\tau',\tau)\,d\tau'\,. \tag{6.59}$$

[6] the same LHS arises if one takes the contour *just above* but now there the RHS is $2\pi i\tilde{G}(\omega)$

The structure (6.59) explains why Green's functions are often called *propagators*

Example The Harmonic Response Curve is the image in the complex $\tilde{G}(\omega)$-plane as ω moves along a of a large semi-circle in the lower half of the complex ω-plane. The curve pursued by ω is $(\mathbb{R} \ni -a < \omega < a) \sqcup \{|\omega| = a\, \Im\omega < 0\}$ for very large $a > 0$. For a stable or *passive* system, for which , moreover $\tilde{G}(\omega)$ never vanishes, the Harmonic Response Curve should have zero net winding number $W(\tilde{G}(\gamma), 0)$ about the origin. On other hand of one, if the semi-circle is in the UHP, the winding number about the origin, $W(\tilde{G}(\gamma), 0)$ counts the number of transients.

Example The damped simple harmonic oscilator

$$\tilde{G}(\omega) = \frac{1}{\omega_0^2 - \omega^2 + \frac{2i\omega}{\tau}} . \tag{6.60}$$

There are two poles in the UHP. Since $\overline{\tilde{G}(\omega)} = \tilde{G}(-\omega)$, the Harmonic Response Curve is reflection symmetric about the real axis. As ω starts from near $-\infty$ it moves from a point near the origin in the second quadrant in a clockwise direction in the UHP intersecting the real axis at $\omega = 0$. It then continues in a clockwise direction until it reaches a point near zero in the third quadrant. As ω moves along a large semi-circle in the LHP we have

$$\tilde{G}(\omega) \approx -\frac{1}{\omega^2} + \dots \tag{6.61}$$

and therefore $\arg \tilde{G}(\omega)$ runs through 2π a small circle centred on the origin in the anti-clockwise sense. Thus the net winding number is zero.

On the other hand, if the large semi-circle is in the UHP, the small circle is traversed in the anti-clockwise sense and the net winding number is 2.

Example The Lorentz-Dirac equation in a simple harmonic potential

$$\frac{d^2y}{dt^2} - \epsilon \frac{d^3y}{dt^3} + \omega_0^2 y = f(t) , \qquad \epsilon > 0 . \tag{6.62}$$

Physically was introduced to describe the motion of an electron in a harmonic potential. The the third derivative term was designed to account for *radiation reaction damping* due to the energy loss by emission of electromagnetic waves by the accelerating electron. However The system is unstable, there are two poles the UHP and one in the LHP. The two poles in the UHP correspond to damping of the undamped oscillations of the oscillator without radiation reaction as expected. The pole in the LHP gives rise to runaway solutions.

If we were to solve the equation by the Fourier method, which assumes that $y(t)$ is bounded for all times, the electron would exhibit *pre-acceleration*: it starts moving before it is hit!. A causally sensible solution can be found using the Laplace transform.

6.8 Laplace Transformations

It often happens that one wishes to consider functions which are not in L^1 or L^2, and hence for which the FT does not exist. One may then turn to the Laplace Transform

Definition In the *space domain*, The Laplace Transform or LT of a real valued function $f(t)$ is, for $\Re s$ sufficiently large that the integral exists

$$\boxed{\mathcal{L}(f(t)) = \hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt} \tag{6.63}$$

In fact if $f(t)$ is *exponentially bounded*

$$|f(t)| < Me^{Nt}\,, M, N > 0\,, \qquad \text{as}\, t \to \infty \tag{6.64}$$

then $\mathcal{L}(f(t)) = \hat{f}(s)$ will exist and is analytic for $\Re s > N$.

For notational purposes it is convenient to restrict attention to functions which vanish for $t < 0$. If $f(t)$ is not in this class, then $H(t)f(t)$ where .

Definition The *Heaviside Function* $H(t) = 0\,. \quad t < 0$, $\quad H(t) = 1\,, \quad t > 0\,.$

The Laplace Transform enjoys the following properties

$$(i) \quad \text{Linearity :} \quad \mathcal{L}(f_1 + f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_1) \tag{6.65}$$

$$(ii) \quad \text{Translation :} \quad \mathcal{L}(H(t-t_o)f(t-t_0)) = e^{-st_0}\mathcal{L}(f(t)) = e^{-st_0}\hat{f}(s) \tag{6.66}$$

$$(iii) \quad \text{Scaling :} \quad \mathcal{L}(f(at)) = \frac{1}{a}\hat{f}(\frac{s}{a}) \tag{6.67}$$

$$(iv) \quad \text{Multiplication :} \quad \mathcal{L}(tf(t)) = -\frac{d\hat{f}}{ds}\,, \quad \mathcal{L}(t^k f(t)) = (-1)^k \frac{d^k \hat{f}}{ds^k} = (-1)^k \hat{f}(s)^{(k)} \tag{6.68}$$

$$(v) \quad \text{Derivation :} \quad \mathcal{L}\left(\frac{df}{dt}\right) = s\hat{f}(s) - f(0) \tag{6.69}$$

The derivation property (6.69 (v)) of the LT is rather different from that of the FT. The FT of $\dot{f}$ depends on its *initial* value. Similarly

$$\mathcal{L}\left(\frac{df}{dt}\right) = s\hat{f}(s) - f(0) \tag{6.70}$$

$$\mathcal{L}\left(\frac{d^2 f}{dt^2}\right) = s^2\hat{f}(s) - sf(0) - f'(0) \tag{6.71}$$

$$\mathcal{L}\left(\frac{d^k f}{dt^k}\right) = s^k\hat{f}(s) - s^{k-1}f(0) - s^{k-2}f'(0) \cdots - f^{n-1}(0) \tag{6.72}$$

For this reason, the LT is much better suited to solving initial value problems than the FT.

Example

$$\mathcal{L}(t^n) = \int_o^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}\,, \qquad n \in \mathbb{N}\,. \tag{6.73}$$

More generally we

Definition **Euler's Gamma Function**

$$\boxed{\Gamma(n) := \int_0^\infty e^{-t}t^{n-1}\, dt\,, \qquad \Re\, n > 0} \tag{6.74}$$

So that $\Gamma(n) = (n-1)! \quad n = 1, 2, \ldots$. If one integrates by parts one finds that

$$\Gamma(n) = (n-1)\Gamma(n-1)\,, \tag{6.75}$$

Also, on substituting $t = \frac{1}{2}u^2$, one finds

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}\,\frac{dt}{\sqrt{t}} = \int_0^\infty e^{\frac{1}{2}u^2}\sqrt{2}\,du = \sqrt{\pi} \implies \Gamma(\frac{3}{2}) = (\frac{3}{2}-1)\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2} = (\frac{1}{2})! \tag{6.76}$$

In fact $\Gamma(z)$ is holomorphic except at $z = -1, -2\ldots$, where it has simple poles. Evidently

$$\boxed{\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{t^{\alpha+1}}\,, \qquad \alpha \neq -1, -2, \ldots} \tag{6.77}$$

6.9 Convolution Property

This is

$$\boxed{\mathcal{L}(f \star g) = \hat{f}(s)\hat{g}(s)} \tag{6.78}$$

and is proved by integration by parts.

Note that if $f(t)$ and $g(t)$ both vanish for $t < 0$ then

$$f \star g = \int_0^x f(t)g(x-t)\,dt \tag{6.79}$$

Example *Euler's complete Beta function*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta\,, \tag{6.80}$$

where we have used the substitution $t = \cos^2\theta$. One sets

$$f(x) = H(x)x^{p-1}\,, \qquad g(x) = H(x)x^{q-1}\,, \tag{6.81}$$

so

$$B(p,q) = f \star g\Big|_{x=1}\,, \tag{6.82}$$

Now

$$\mathcal{L}\big(H(x)x^{p-1}\big) = \frac{\Gamma(p)}{s^p}\,, \qquad \mathcal{L}\big(H(x)x^{p-1}\big) = \frac{\Gamma(q)}{s^q}\,, \qquad \Rightarrow \qquad \mathcal{L}\big(f \star g\big) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,\frac{\Gamma(p+q)}{s^{p+q}}\,, \tag{6.83}$$

hence

$$f \star g = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}x^{p+q-1} \tag{6.84}$$

Setting $x = 1$ gives

$$\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta = \frac{1}{2}B(p,q) = \frac{1}{2}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,. \tag{6.85}$$

A special case is the Wallis integral

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = 2B(n+\frac{1}{2},\frac{1}{2}). \tag{6.86}$$

Using (6.75) and (6.76) one may check that one gets the same answer as using the contour integral (5.14).

Example *Application of Laplace and Fourier transformations to Probability Theory*

Recall that if a random variable X has probability distribution $P(x)dx$ then the expectation value of an observable $f(x)$ with respect to that distribution is

$$\mathbb{E}_P(f(x)) = \int f(x)P(x)dx. \tag{6.87}$$

If X and Y are random variables independently distributed with probability distributions $P_1(x)dx$ and $P_2(y)dy$, Then

$$\begin{aligned} \mathbb{E}\big(f(x)g(y)\big) &= \int\int f(x)g(y)P_1(x)P_2(y)\,dxdy & (6.88)\\ &= \mathbb{E}_{P_1}(f(x))\,\mathbb{E}_{P_2}(g(y)) & (6.89) \end{aligned}$$

If a random variable X has probability distribution $P_1(x)dx$ and a random variable Y has an independent probability distribution $P_2(y)dy$ then their joint probability distribution is

$$P_1(x)P_2(y)dxdy = P_1(x)P_2(z-x)dxdz = P_1(z-y)P_2(y)dydz\,. \tag{6.90}$$

where $z = x+y$. Thus the sum $Z = X+Y$ has probability distribution $P(z)$ where

$$P(z)dz = \int P_1(x)P_2(z-x)\,dx = \int P_1(z-y)P_2(y)\,dy\,. \tag{6.91}$$

In other words $P(z)$ is given by the convolution

$$P(z) = P_1 \star P_2(z) = P_2 \star P_1(z)\,. \tag{6.92}$$

We define

$$\begin{aligned} \mathbb{E}_P\big(e^{-ikx}\big) &= \int_{-\infty}^{\infty} P(x)e^{-ikx}\,dx & (6.93)\\ &= \mathcal{F}\big(P(x)\big) & (6.94)\\ &= \tilde{P}(k)\,. & (6.95) \end{aligned}$$

so that the moments are given by

$$\mathbb{E}_P(x^n) = \frac{1}{(-i)^n n!}\frac{d^n\tilde{P}}{dk^n}\Big|_{k=0} \tag{6.96}$$

For two independently distributed random variables X,Y the sum has characteristic function

$$\begin{aligned} \mathbb{E}_{P_1\star P_2}\big(e^{-ik(x+y)}\big) &= \mathbb{E}_{P_1}\big(e^{-ikx}\big)\mathbb{E}_{P_2}\big(e^{-iky}\big) & (6.97)\\ &= \tilde{P}_1(k)\,\tilde{P}_2(k) & (6.98)\\ &= P_1 \tilde{\star} P_2(k)\,. & (6.99) \end{aligned}$$

As an *example*, suppose that $P_1(x), P_2(x)$ are Gaussians with mean zero. The FT of a Gaussian is a Gaussian centred on zero and the product of two such Gaussians is a Gaussian centred on zero. It follows that the sum of two Gaussian random variables is a Gaussian random variables. Writing this out in formulae allows one to calculate its standard deviation in terms of those of $P_1(x)$ and $P_2(x)$. If the Gaussians do not have mean zero one can repeat the exercise using the shift property of the FT.

In the above we assumed the sample space to be $\mathbb{R}$ the entire real line. However we could consider it to be the positive real axis $\mathbb{R}_+$. Then the moment generating function is the Laplace transform

$$\begin{aligned} \mathbb{E}_P\big(e^{-sx}\big) &= \int_0^\infty P(x)e^{-sx}\,dx && (6.100)\\ &= \mathcal{L}\big(P(x)\big) = \hat{P}(s)\,. && (6.101) \end{aligned}$$

6.10 Inverse Laplace Transform and Bromwich contour

Proposition If $\hat{f}(s)$ is holomorphic for $\Re s > c > 0$, then

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{c+i\infty}^{c+i\infty} e^{st}\hat{f}(s)\,ds\,.} \tag{6.102}$$

Proof

If $t < 0$ we drag the so-called *Bromwich contour* $\Re s = c\,;\ -\infty < v = \Im s < \infty$ to the right. In the limit $\Re s \to \infty$ $e^{st}\hat{f}(s) \to 0$ and so $f(t) = 0\,, t < 0$.

For $t > 0$, setting $s = u + iv, u, v \in \mathbb{R}$,

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{tu}e^{iv}\Big(\int_0^\infty e^{-st'}f(t')\,dt'\Big)\,idv && (6.103)\\ &= \int_{-\infty}^{\infty} e^{u(t-t')}f(t')\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iv(t-t')}\,dv\Big)\,dt' && (6.104)\\ &= \int_1^\infty e^{u(t-t')}f(t')\delta(t-t')\,dt' && (6.105)\\ &= f(t)\,. && (6.106) \end{aligned}$$

Example $\hat{f}(s) = \frac{1}{(s-2)^2}$ We need $c > 2$. For $t > 0$ we drag the Bromwich contour to the *left* picking up a pole contribution at $s = 2$. The remaining integral vanishes in the limit $c \to -\infty$ and so

$$f(t) = \mathrm{Res}[\frac{e^{st}}{(s-2)^2}, 2] = te^{2t}\,, \qquad t > 0\,. \tag{6.107}$$

6.11 Solution of initial value problems

Example A forced *unstable* oscillator

$$\ddot{y} - 3\dot{y} + 2y = 4e^{2t}\,, \qquad y(0) = -3\,, \quad \dot{y}(0) = 5\,. \tag{6.108}$$

$$\implies \mathcal{L}(\ddot{y} - 3\dot{y} + 2y)(s) = \mathcal{L}(4e^{2t})(s) = \frac{4}{s-2}\,, \tag{6.109}$$

$$\mathcal{L}(\dot{y}(t))(s) = \mathcal{L}(y(t)(s) - y(0) = \hat{y}(s) + 3 \tag{6.110}$$

$$\mathcal{L}(\ddot{y}(t)) = s^2\mathcal{L}(y(t)(s) - sy(0) - \dot{y}(0)) = s^2\hat{y}(s) + 3s - 5\,, \tag{6.111}$$

$$\implies \mathcal{L}(\ddot{y} - 3\dot{y} + 2y) = (s-1)(s-2)\hat{y}(s) + 3s - 14 = \frac{4}{s-2} \tag{6.112}$$

$$\implies \hat{y}(s) = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2} \tag{6.113}$$

$$\implies y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}\,. \tag{6.114}$$

It is quite impossible to solve this using the FT since the relevant FT's don't exist.

Example A system

$$\dot{y} = y - x\,, \qquad y(0) = 1\,, \tag{6.115}$$

$$\dot{x} = 5y - 3x\,, \quad x(0) = 2\,. \tag{6.116}$$

$$\implies s\hat{y}(s) - 1 = \hat{y}(s) - \hat{x}(s)\,, \tag{6.117}$$

$$s\hat{x}(s) - 2 = 5\hat{y} - 3\hat{x}\,. \tag{6.118}$$

$$\text{i.e.} \qquad \begin{pmatrix} s-1 & 1 \\ 5 & -(s+3) \end{pmatrix} \begin{pmatrix} \hat{y}(s) \\ \hat{x}(s) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{6.119}$$

$$\implies \begin{pmatrix} \hat{y}(s) \\ \hat{x}(s) \end{pmatrix} = -\frac{1}{s^2+2s+2} \begin{pmatrix} -(s+3) & -1 \\ 5 & (s-1) \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{6.120}$$

$$\implies \hat{y}(s) = \frac{s+1}{s^2+2s+2} \tag{6.121}$$

$$\implies y(t) = e^{-t}\cos t \tag{6.122}$$

$$\hat{y}(s) = \frac{2s+3}{s^2+2s+2} \tag{6.123}$$

$$\implies x(t) = e^{-t}\big(2\cos t + \sin t\big)\,. \tag{6.124}$$

6.12 Linear Integral Equations and feedback loops

Suppose we want to solve

$$y(t) = f(t) - \int_0^t K(t-\tau)y(\tau)\,d\tau \tag{6.125}$$

Taking a Laplace transformation gives

$$\hat{y}(s) = \hat{f}(s) - \hat{K}(s)\hat{y}(s)\,, \qquad \Leftrightarrow \qquad \hat{y}(s) = \frac{\hat{f}(s)}{1+\hat{K}(s)}\,. \tag{6.126}$$

Thus

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}\hat{f}(s)}{1+\hat{K}(s)}\,ds\,. \tag{6.127}$$

More generally one could consider the system

$$Dy(t) = f(t) - \int_0^t K(t-\tau)y(\tau)\,d\tau\,, \tag{6.128}$$

where D is some differential operator whose inverse has LT $\hat{G}(s)$. Then

$$\frac{\hat{y}(s)}{\hat{G}(s)} = \hat{f}(s) - \hat{K}(s)\hat{y}(s)\,, \qquad \Leftrightarrow \qquad \hat{y}(s) = \frac{\hat{G}(s)}{1+\hat{G}(s)\hat{K}(s)}\,\hat{f}(s)\,. \tag{6.129}$$

and thus

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{\hat{G}(s)\hat{f}(s)}{1+\hat{G}(s)\hat{K}(s)}\,ds\,. \tag{6.130}$$

In the language of systems theory one says that the output due to $f(t)$ from the integral operator with kernel $K(t-\tau)$ is fed back and subtracted from the input. If the sign were reversed, we would have positive feed back.

The same problem can be tacked using the Fourier Transform.

Example *The stability of Feedback: the Nyquist stability criterion*

Using the FT, the feedback equation would be

$$\frac{1}{\tilde{G}(\omega)}\tilde{y}(\omega) = \tilde{f}(\omega) - \tilde{K}(\omega)\tilde{y}(\omega) \tag{6.131}$$

so

$$\tilde{y}(\omega) = \frac{\tilde{G}(\omega)}{1+\tilde{G}(\omega)\tilde{K}(\omega)}\,\tilde{f}(\omega)\,. \tag{6.132}$$

The quantity $\frac{\tilde{G}(\omega)}{1+\tilde{G}\tilde{K}(\omega)}$ is often referred to as the *closed loop gain* and $\tilde{G}\tilde{K}$ as the *open loop gain*.

The principal of a *feedback amplifier* is to choose $\tilde{G}(\omega)\tilde{K}(\omega) >> 1$ so that

$$\tilde{y}(\omega) \approx \frac{\tilde{f}(\omega)}{\tilde{K}(\omega)}\,. \tag{6.133}$$

Now it is technically difficult to make $\tilde{G}(\omega)$ large and independent of frequency ω, however is it technically easy to make $\tilde{K}(\omega)$ small and independent of ω. This allows high amplificaton with a high degree of fidelity.

If the system without feedback, i.e. when $\tilde{K}(\omega) = 0$, is stable and $\tilde{G}(\omega)$ never vanishes, then the system with feedback will be stable if the open loop gain Harmonic Response Curve given by $\tilde{G}(\omega)\tilde{K}(\omega)$ is never zero and has zero winding number about $(-1,0)$.

6.13 Use of Laplace Transform to solve P.D.E.'s

Example The diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x>0\,,t\geq o\,, \tag{6.134}$$

$$(i) \qquad u(x,0) = 0\,, \tag{6.135}$$

$$(ii) \qquad \lim x\uparrow\infty \quad u(x,t) = 0\,, \tag{6.136}$$

$$(iii) \qquad u(0,t) = \text{constant} = \text{u}_0\,. \tag{6.137}$$

We take the LT with respect to t using (i) and then (ii) to get

$$s\hat{u}(x,s) = \frac{d^2\hat{u}(x,s)}{dx^2} \implies \hat{u} = A(s)e^{-\sqrt{s}x}\,. \tag{6.138}$$

Set $x=0$ and use (iii) to get

$$A(s) = \mathcal{L}(u_o)\,, \implies \hat{u}(x,s) = \frac{u_0}{s}e^{-\sqrt{s}x}\,. \tag{6.139}$$

$$\implies u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{u_0}{s}e^{(-\sqrt{s}x+st)}\,ds \tag{6.140}$$

We take $\sqrt{s}$ to be positive on the positive real axis and cut the plane along the negative real axis. The integrand thus has a branch cut along the negative real axis and a singularity at the branch point and simple pole at the origin. For $t>0$ deform the Bromwich contour to a Keyhole contour. The contribution from the pole is u_0. The contributions from the cut are

$$\text{above the cut} \quad u(x,t) = \frac{1}{2\pi i}\int_0^{-\infty}\frac{u_0}{s}e^{(-ix\sqrt{-s}+st)}\,ds \tag{6.141}$$

$$\text{below the cut} \quad = \frac{1}{2\pi i}\int_{-\infty}^{0}\frac{u_0}{s}e^{(+ix\sqrt{-s}+st)}\,ds \tag{6.142}$$

$$\tag{6.143}$$

Thus, setting $u^2=-s$ and $v^2=\frac{x}{\sqrt{t}}$,

$$u(x,t) = -\frac{u_0}{\pi i}\int_{-\infty}^{\infty}e^{iux-tu^2}\,\frac{du}{u} \tag{6.144}$$

$$\implies \frac{du(x,t)}{dx} = -\frac{u_0}{\pi}\int_{-\infty}^{\infty}e^{iux-tu^2}\,du \tag{6.145}$$

$$= -\frac{u_0}{\pi}\int_{-\infty}^{\infty}e^{-t(u-\frac{ix}{2t})^2-\frac{x^2}{4t}}\,du \tag{6.146}$$

$$= -\frac{u_0}{\sqrt{\pi}}\frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}} \tag{6.147}$$

$$\implies u(x,t) = -u_o\frac{2}{\sqrt{\pi}}\int_o^{\frac{x}{2\sqrt{t}}}e^{-v^2}\,dv + \text{constant} \tag{6.148}$$

$$= u_o\left(1-\ \text{erf}(\frac{x}{2\sqrt{t}})\right)\,, \tag{6.149}$$

where

Definition The error function $\mathrm{erf}(x)$ is

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2}\, dv\,. \tag{6.150}$$

So that $\mathrm{erf}(0) = 0$, $\mathrm{erf}(\infty) = 1$.

Example The diffusion equation may also be solved by noting that

$$\mathcal{L}\left(\frac{e^{-b/4t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}} e^{-\sqrt{bs}}\,, \qquad b > 0\,, \tag{6.151}$$

which follows from

$$\int_0^\infty e^{-\frac{1}{2}(au^2+\frac{b}{u^2})}\, du = \frac{1}{2}\int_\infty^\infty e^{-\frac{1}{2}(au^2+\frac{b}{u^2})}\, du = \sqrt{\frac{\pi}{2a}} e^{-\sqrt{ab}} \tag{6.152}$$

which one can prove by substituting $v = u\sqrt{a} - \frac{\sqrt{b}}{u}$.

Example Setting $b = x^2$ one deduces that

$$G(x - x', t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x')2}{4t}} \tag{6.153}$$

solves the diffusion equation for $-\infty < x < +\infty$, $t > 0$ such that

$$\lim_{|x|\uparrow\infty} G(x - x', t) = 0\,, \tag{6.154}$$

$$\lim_{t\downarrow 0} G(x - x', t) = \delta(x - x')\,. \tag{6.155}$$

6.14 Solving the the wave equation using the Laplace Transform

Example The Green function $G(\mathbf{x} - \mathbf{x}', t - t')$ satisfies

$$-\nabla^2 G + \frac{\partial^2 G}{\partial t^2} = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}'),. \tag{6.156}$$

By translation invariance we set $\mathbf{x}' = 0 = t'$. Take the Laplace transform with respect to t. One has

$$\delta(t) = H'(t) \Rightarrow \mathcal{L}(\delta(t)) = s\mathcal{L}(H(t)) = s\frac{1}{s} \Rightarrow \mathcal{L}(\delta(t)) = 1\,. \tag{6.157}$$

Thus

$$-\nabla^2 \hat{G}(\mathbf{x}, s) + s^2 \hat{G}(\mathbf{x}, s) = \delta(\mathbf{x})\,. \tag{6.158}$$

The solution which decays at infinity is

$$\hat{G}(\mathbf{x}, s) = G(r, s) = \frac{1}{4\pi r} e^{-sr}\,, \qquad r = |\mathbf{x}|\,. \tag{6.159}$$

Thus

$$G(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{4\pi r} e^{-s(r-t)}\, ds \tag{6.160}$$

If $r > t$ we can move the Bromwich contour to the left and hence

$$G(r, t) = 0\,. \tag{6.161}$$

On the other hand, if $r < t$ we set $s = c + iv$, $ds = idv$ and recall that $f(x)\delta(x) = f(0)\delta(x)$, we find

$$\begin{aligned} G(r,t) &= \frac{1}{4\pi r} \int_{\infty}^{\infty} e^{-iv(r-t)}\, e^{-c(r-t)} && (6.162) \\ &= \frac{1}{4\pi r} e^{-c(r-t)} \delta(r-t) && (6.163) \\ &= \frac{1}{4\pi r} \delta(r-t)\,. && (6.164) \end{aligned}$$

Inverting the translation we find the *Retarded Green's function*

$$\begin{aligned} G(\mathbf{x}-\mathbf{x}', t-t') &= 0\,, \qquad t - t' < |\mathbf{x} - \mathbf{x}'| && (6.165) \\ &= \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} \delta(|\mathbf{x} - t + t')\,, \qquad t - t' > |\mathbf{x} - \mathbf{x}'|\mathbf{x}'|\,. && (6.166) \end{aligned}$$